Sobolev functions and smooth change of coordinates

Teorema 1. *Let Ω and $\widetilde{\Omega}$ be two open sets in $\mathbb{R}^d$. Let*

$$\Phi = (\Phi_1, \dots, \Phi_d) : \Omega \to \widetilde{\Omega}$$

be a map of class C^1 on Ω with inverse

$$\Psi = (\Psi_1, \dots, \Psi_d) : \widetilde{\Omega} \to \Omega$$

of class C^1 on $\widetilde{\Omega}$. Assume moreover that

$$\sup_{x\in\Omega}\left\{\sum_{j=1}^{d}|\partial_j\Phi(x)|\right\} + \sup_{y\in\widetilde{\Omega}}\left\{\sum_{j=1}^{d}|\partial_j\Psi(y)|\right\} < +\infty.$$

Let $p \in (1, +\infty)$. Then, the following holds:

(i) *if $\Omega, \widetilde{\Omega}$ are open sets and if $u \in W_0^{1,p}(\widetilde{\Omega})$, then $u \circ \Phi \in W_0^{1,p}(\Omega)$;*

(ii) *if Ω and $\widetilde{\Omega}$ are bounded open sets, and if $\widetilde{\Omega}$ is of class C^1, then for every $u \in W^{1,p}(\widetilde{\Omega})$ we have $u \circ \Phi \in W^{1,p}(\Omega)$.*

Moreover, in both cases we have the formula

$$\nabla(u\circ\Phi) := J\Phi[\nabla u(\Phi)] \quad \text{Lebesgue almost-everywhere on} \quad \Omega\,,$$

where we recall the notation

$$\nabla u := \begin{pmatrix}\partial_1 u\\ \vdots\\ \partial_d u\end{pmatrix} \qquad \text{and} \qquad J\Phi := \begin{pmatrix}\partial_1\Phi_1 & \dots & \partial_1\Phi_d\\ \vdots & \ddots & \vdots\\ \partial_d\Phi_1 & \dots & \partial_d\Phi_d\end{pmatrix}. \tag{1}$$

Osservazione 2. *We recall that for a generic matrix*

$$A = \begin{pmatrix}a_{11} & \dots & a_{1d}\\ \vdots & \ddots & \vdots\\ a_{d1} & \dots & a_{dd}\end{pmatrix},$$

we use the notation

$$\|A\|_2^2 := \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}^2,$$

and we have that

$$|Av| \le \|A\|_2|v| \quad \text{for all} \quad v \in \mathbb{R}^d,$$

where $|v|$ is the usual euclidean norm of the vector $v \in \mathbb{R}^d$.

Proof of Theorem 1. We first notice that if $u \in C_c^\infty(\widetilde{\Omega})$, then $u \circ \Phi \in C_c^1(\Omega)$ and we have the usual chain rule for smooth functions

$$\nabla(u\circ\Phi)(x) = J\Phi(x)[\nabla u(\Phi(x))] \quad \text{for all} \quad x \in \Omega.$$

We first prove (i). Let $u \in W_0^{1,p}(\widetilde{\Omega})$. By the definition of $W_0^{1,p}(\widetilde{\Omega})$, there is a sequence u_n of functions in $C_c^\infty(\widetilde{\Omega})$ that converges to u strongly in $W_0^{1,p}(\widetilde{\Omega})$ and pointwise almost-everywhere in $\widetilde{\Omega}$. Using again the change of variables $y = \Phi(x)$, we have

$$\int_\Omega |u_n(\Phi(x)) - u_m(\Phi(x))|^p\,dx = \int_{\widetilde{\Omega}} |u_n(y) - u_m(y)|^p\,|\det(D\Psi(y))|\,dy,$$

for all $m, n \in \mathbb{N}$. As a consequence, $(u_n \circ \Phi)_{n\ge1}$ is a Cauchy sequence in $L^p(\Omega)$, so it converges to some function $v \in L^p(\Omega)$. Now since $u_n \circ \Phi$ converges pointwise almost-everywhere in Ω to $u \circ \Phi$, we have that $u \circ \Phi = v$, so

$$u_n \circ \Phi \to u \circ \Phi \quad \text{strongly in} \quad L^p(\Omega).$$

In order to prove the convergence of the gradients of $u_n \circ \Phi$, we proceed analogously.

Indeed, applying the estimate from Remark 2 to the $d \times d$ matrices $A = J\Phi(x)$ for $x \in \Omega$ we have

$$\begin{aligned}\int_\Omega |J\Phi(x)\nabla u_n(\Phi(x)) - J\Phi(x)\nabla u_m(\Phi(x))|^p\,dx &\le \sup_{x\in\Omega}\|J\Phi(x)\|_2^p \int_\Omega |\nabla u_n(\Phi(x)) - \nabla u_m(\Phi(x))|^p\,dx \\ &= \sup_{x\in\Omega}\|J\Phi(x)\|_2^p \int_{\widetilde\Omega} |\nabla u_n(y) - \nabla u_m(y)|^p\,|\det(J\Psi(y))|\,dy \\ &\le \sup_{x\in\Omega}\|J\Phi(x)\|_2^p \sup_{y\in\widetilde\Omega}|\det(J\Psi(y))| \int_{\widetilde\Omega} |\nabla u_n - \nabla u_m|^p,\end{aligned}$$

where we recall that $\Psi : \widetilde\Omega \to \Omega$ is the inverse of Φ. Now, by the choice of the sequence u_n we have that

$$\partial_j u_n \to \partial_j u \quad \text{strongly in} \quad L^p(\widetilde\Omega),$$

for all $j = 1, \dots, d$. Thus, $\big(J\Phi\nabla u_n(\Phi)\big)_{n\ge1}$ is a Cauchy sequence in $L^p(\Omega;\mathbb{R}^d)$. Since we have the pointwise convergence $J\Phi(x)[\nabla u_n(\Phi(x))] \to J\Phi(x)[\nabla u(\Phi(x))]$ for almost-every $x \in \Omega$, we get that $J\Phi[\nabla u_n(\Phi)]$ converges strongly to $J\Phi[\nabla u(\Phi)]$ in $L^p(\Omega;\mathbb{R}^d)$. Thus, $u\circ\Phi$ is in $W^{1,p}_0(\Omega)$ and the weak gradient $\nabla(u\circ\Phi)$ can be computed via the chain rule formula

$$\nabla(u\circ\Phi) = J\Phi[\nabla u(\Phi)].$$

We next prove claim (ii). Let $u \in W^{1,p}(\widetilde\Omega)$. Since, by hypothesis, $\widetilde\Omega$ is a bounded open set with C^1-regular boundary, we have that there is a sequence u_n of functions in $C^1(\widetilde\Omega)$ that converges to u strongly in $W^{1,p}(\widetilde\Omega)$. By extracting a subsequence, we can suppose that

$$u_n(y) \to u(y) \qquad \text{and} \qquad \partial_j u_n(y) \to \partial_j u(y),$$

for Lebesgue almost-every $y \in \widetilde\Omega$ and every $j = 1, \dots, d$. The rest of the proof is identical to the proof of (i). First, using the change of variables $y = \Phi(x)$, we have

$$\int_\Omega |u_n(\Phi(x)) - u_m(\Phi(x))|^p\,dx = \int_{\widetilde\Omega} |u_n(y) - u_m(y)|^p\,|\det(D\Psi(y))|\,dy,$$

for all $m, n \in \mathbb{N}$. This implies that $(u_n\circ\Phi)_{n\ge1}$ is a Cauchy sequence in $L^p(\Omega)$, which together with the pointwise almost-everywhere convergence of $u_n\circ\Phi$ to $u\circ\Phi$ gives that

$$u_n\circ\Phi \to u\circ\Phi \quad \text{strongly in} \quad L^p(\Omega).$$

For the strong convergence of the gradients we use again the estimate

$$\int_\Omega |J\Phi(x)\nabla u_n(\Phi(x)) - J\Phi(x)\nabla u_m(\Phi(x))|^p\,dx \le \sup_{x\in\Omega}\|J\Phi(x)\|_2^p \sup_{y\in\widetilde\Omega}|\det(J\Psi(y))| \int_{\widetilde\Omega} |\nabla u_n - \nabla u_m|^p.$$

Now, the strong $L^p(\widetilde\Omega;\mathbb{R}^d)$ convergence of ∇u_n to ∇u implies that $J\Phi[\nabla u_n(\Phi)]$ is a Cauchy sequence in $L^p(\Omega;\mathbb{R}^d$, while from the pointwise convergence $J\Phi(x)[\nabla u_n(\Phi(x))] \to J\Phi(x)[\nabla u(\Phi(x))]$ for almost-every $x \in \Omega$, we get that $J\Phi[\nabla u_n(\Phi)]$ converges strongly in $L^p(\Omega;\mathbb{R}^d)$ to $J\Phi[\nabla u(\Phi)]$ in $L^p(\Omega;\mathbb{R}^d)$. Thus, we get that $u\circ\Phi$ is in $W^{1,p}(\Omega)$ and the weak gradient $\nabla(u\circ\Phi)$ can be computed via the chain rule formula (1). $\square$