The p-capacity in $\mathbb{R}^d$

Definition

Definition 1 (p-capacity in $\mathbb{R}^d$.)**.** *Let $p \in [1, +\infty)$ and $d \geq 1$. For every set $A \subset \mathbb{R}^d$, we define the p-capacity of A in $\mathbb{R}^d$ as follows:*

$$\mathrm{cap}_p(A) = \inf\Big\{\int_{\mathbb{R}^d} |\nabla u|^p\,dx \ :\ u \in W^{1,p}(\mathbb{R}^d),\ u \geq 1 \quad \textit{almost-everywhere in a neighborhood of } A\Big\}.$$

Remark 2. *We notice that the quantity $\mathrm{cap}_p(A)$ above is defined for any set $A \subset \mathbb{R}^d$. In particular, A can be taken unbounded or not Lebesgue measurable.*

Proposition 3. *Let $p \in [1, +\infty)$ and $d \geq 1$.*

(i) *The p-capacity of any set $A \subset \mathbb{R}^d$ can be computed as follows*

$$\mathrm{cap}_p(A) = \inf\Big\{\int_{\mathbb{R}^d} |\nabla u|^p\,dx \ :\ u \in W^{1,p}(\mathbb{R}^d),\ 0 \leq u \leq 1 \ \textit{ on } \mathbb{R}^d,\ u = 1 \ \textit{ a.e. in a neighborhood of } A\Big\};$$

(ii) *For any open set $A \subset \mathbb{R}^d$ we have*

$$\mathrm{cap}_p(A) = \inf\Big\{\int_{\mathbb{R}^d} |\nabla u|^p\,dx \ :\ u \in W^{1,p}(\mathbb{R}^d),\ 0 \leq u \leq 1 \ \textit{ on } \mathbb{R}^d,\ u = 1 \ \textit{ on } A\Big\}.$$

Proof. In order to prove claim (i), we recall that, for every $u \in W^{1,p}(\mathbb{R}^d)$, we have $0 \vee u \wedge 1 \in W^{1,p}(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} |\nabla(0 \vee u \wedge 1)|^p\,dx = \int_{\{0<u<1\}} |\nabla u|^p\,dx \leq \int_{\mathbb{R}^d} |\nabla u|^p\,dx.$$

Claim (ii) is a direct consequence of claim (i). □

Lemma 4 (Monotonicity with respect to inclusion)**.** *Let $p \in [1, +\infty)$ and $d \geq 1$. Suppose that A_1 and A_2 are two subsets of $\mathbb{R}^d$. If $A_1 \subset A_2$, then $\mathrm{cap}_p(A_1) \leq \mathrm{cap}_p(A_2)$.*

Proof. This follows immediately from the definition of $\mathrm{cap}_p(A_1)$ and $\mathrm{cap}_p(A_2)$. □

Corollary 5. *Let $p \in [1, +\infty)$ and $d \geq 1$. If A is a bounded subset of $\mathbb{R}^d$, then $\mathrm{cap}_p(A) < +\infty$.*

Proof. Let B_R be a ball containing A and let $\phi_R : \mathbb{R}^d \to \mathbb{R}$ be a smooth function with bounded support in $\mathbb{R}^d$ such that $\phi_R \equiv 1$ on $\mathbb{R}^d$. By definition,

$$\mathrm{cap}_p(B_R) \leq \int_{\mathbb{R}^d} |\nabla \phi_R|^p\,dx.$$

On the other hand, by the monotoniciy of the capacity,

$$\mathrm{cap}_p(A) \leq \mathrm{cap}_p(B_R),$$

which proves the claim. □

The case $p \geq d$.

In this section we show that, when $p \geq d$, the p-capacity is trivial in $\mathbb{R}^d$, that is, the p-capacity of every set in $\mathbb{R}^d$ is 0 or $+\infty$. We will also show that all the bounded sets in $\mathbb{R}^d$ have zero p-capacity. We start with the following lemma.

Lemma 6. *Let $d \geq 2$ and $p \in (1, +\infty)$. For every $0 < r < R < +\infty$, there is a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that*

- $\varphi \in W^{1,p}_0(B_R)$;
- $0 \leq \varphi \leq 1$ *on* $\mathbb{R}^d$;
- $\varphi = 1$ *on* B_r;

and such that

$$\int_{\mathbb{R}^d} |\nabla\varphi|^p\,dx = \begin{cases} d\omega_d\left(\dfrac{\frac{p-d}{p-1}}{R^{\frac{p-d}{p-1}} - r^{\frac{p-d}{p-1}}}\right)^{p-1} & \textit{if} \quad p \neq d; \\[2ex] d\omega_d\dfrac{1}{(\ln R - \ln r)^{d-1}} & \textit{if} \quad p = d. \end{cases}$$

Proof. Consider the continuous function $\phi : (0, +\infty) \to \mathbb{R}$ defined by

$$\phi(t) = 1 \quad \text{for} \quad t \in (0, r),$$
$$\phi(t) = 0 \quad \text{for} \quad t \in (R, +\infty),$$
$$\partial_t(t^{d-1}(-\phi'(t))^{p-1}) = 0 \quad \text{for} \quad t \in (r, R).$$

One can easily integrate the last identity getting that ϕ' is of the form

$$-\phi'(t) = ct^{-\frac{d-1}{p-1}},$$

for some constant c. We will next integrate again in t. We consider two cases.
Case 1. $p \neq d$. When $p \neq d$, ϕ is of the form

$$\phi(t) = at^{1-\frac{d-1}{p-1}} + b,$$

for some real constants a and b. Now, the boundary conditions

$$ar^{1-\frac{d-1}{p-1}} + b = 1 \quad \text{and} \quad aR^{1-\frac{d-1}{p-1}} + b = 0$$

imply

$$a\Big(r^{1-\frac{d-1}{p-1}} - R^{1-\frac{d-1}{p-1}}\Big) = 1$$

so that

$$a = \frac{1}{r^{1-\frac{d-1}{p-1}} - R^{1-\frac{d-1}{p-1}}} \quad \text{and} \quad b = -\frac{R^{1-\frac{d-1}{p-1}}}{r^{1-\frac{d-1}{p-1}} - R^{1-\frac{d-1}{p-1}}}.$$

Now, we compute

$$\phi'(t) = \frac{1-\frac{d-1}{p-1}}{r^{1-\frac{d-1}{p-1}} - R^{1-\frac{d-1}{p-1}}} t^{-\frac{d-1}{p-1}}$$

and setting

$$\varphi(x) = \phi(|x|)$$

we get

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nabla\varphi|^p\,dx &= d\omega_d \int_r^R t^{d-1}|\phi'(t)|^p\,dt \\
&= d\omega_d \left|\frac{1-\frac{d-1}{p-1}}{r^{1-\frac{d-1}{p-1}} - R^{1-\frac{d-1}{p-1}}}\right|^p \int_r^R t^{d-1}t^{-p\frac{d-1}{p-1}}\,dt \\
&= d\omega_d \left|\frac{1-\frac{d-1}{p-1}}{r^{1-\frac{d-1}{p-1}} - R^{1-\frac{d-1}{p-1}}}\right|^p \int_r^R t^{(d-1)(1-\frac{p}{p-1})}\,dt \\
&= d\omega_d \left|\frac{1-\frac{d-1}{p-1}}{r^{1-\frac{d-1}{p-1}} - R^{1-\frac{d-1}{p-1}}}\right|^p \int_r^R t^{-\frac{d-1}{p-1}}\,dt = d\omega_d \left|\frac{1-\frac{d-1}{p-1}}{r^{1-\frac{d-1}{p-1}} - R^{1-\frac{d-1}{p-1}}}\right|^{p-1}.
\end{aligned}$$

Case 2. $p = d$. Integrating in t, we get that ϕ is of the form

$$\phi(t) = a\ln t + b,$$

for some real constants a and b. Now, the boundary conditions

$$a\ln r + b = 1 \quad \text{and} \quad a\ln R + b = 0$$

imply

$$a\Big(\ln r - \ln R\Big) = 1$$

so that

$$a = -\frac{1}{\ln R - \ln r} \quad \text{and} \quad b = \frac{\ln R}{\ln R - \ln r}.$$

Now, we compute

$$\phi'(t) = -\frac{1}{\ln R - \ln r}\frac{1}{t}$$

and setting

$$\varphi(x) = \phi(|x|)$$

we get

$$\int_{\mathbb{R}^d} |\nabla\varphi|^d \, dx = d\omega_d \int_r^R t^{d-1} |\phi'(t)|^d \, dt$$
$$= d\omega_d \frac{1}{(\ln R - \ln r)^d} \int_r^R t^{d-1} t^{-d} \, dt$$
$$= d\omega_d \frac{1}{(\ln R - \ln r)^d} \int_r^R t^{-1} \, dt = \frac{d\omega_d}{(\ln R - \ln r)^{d-1}}.$$

□

Proposition 7 (The bounded sets have zero capacity, when $p \geq d$)**.** *Let $d \geq 2$ and $p \geq d$. Then, for every bounded set $A \subset \mathbb{R}^d$, we have that* $\mathrm{cap}_p(A) = 0$.

Proof. Since A is bounded, we get that

$$A \subset B_r$$

for some ball $B_r \subset \mathbb{R}^d$. The monotonicity of cap_p implies that

$$\mathrm{cap}_p(A) \leq \mathrm{cap}_p(B_r),$$

so it is sufficient to prove that $\mathrm{cap}_p(B_r) = 0$. In order to do so, for every $R > r$ consider the function ϕ_R given by Lemma 6. Then,

$$\mathrm{cap}_p(B_r) \leq \int_{\mathbb{R}^d} |\nabla\varphi|^p \, dx = \begin{cases} d\omega_d \left(\dfrac{\frac{p-d}{p-1}}{R^{\frac{p-d}{p-1}} - r^{\frac{p-d}{p-1}}} \right)^{p-1} & \text{if} \quad p > d; \\ d\omega_d \dfrac{1}{(\ln R - \ln r)^{d-1}} & \text{if} \quad p = d. \end{cases}$$

In both cases, by sending $R \to +\infty$, we get that $\mathrm{cap}_p(B_r) = 0$. □

Proposition 8 (The unbounded sets have infinite capacity, when $p > d$)**.** *Let $d \geq 2$ and $p > d$. Let $A \subset \mathbb{R}^d$ be an unbounded set in $\mathbb{R}^d$. Then, there are no functions $u \in W^{1,p}(\mathbb{R}^d)$ such that $u = 1$ on A. In particular,* $\mathrm{cap}_p(A) = +\infty$.

Proof. Let A be an unbounded set. Then, there is a sequence of points $a_n \in A$ such that

$$|a_n - a_m| \geq 2 \quad \text{for all} \quad n > m \geq 1.$$

Thanks to the Morrey's Lemma, we have that if $u \in W^{1,p}(\mathbb{R}^d)$, then u is continuous and

$$|u(x) - u(y)| \leq C|x - y|^\alpha,$$

where C is a constant depending only on d, p, and the norm $\|\nabla u\|_{L^p(\mathbb{R}^d)}$. In particular, since $u(a_n) = 1$, we can find a radius $\rho \in (0, 1)$ such that

$$u \geq \frac{1}{2} \quad \text{on} \quad B_\rho(a_n).$$

Since the balls $B_\rho(a_n)$ are mutually disjoint, this is a contradiction with the fact that $u \in L^p(\mathbb{R}^d)$. □

Corollary 9 (The capacity of a set when $p > d$)**.** *Let $p > d \geq 2$. Then, for every set $A \subset \mathbb{R}^d$ we have:*

$$\mathrm{cap}_p(A) = \begin{cases} 0 & \textit{if } A \textit{ is bounded,} \\ +\infty & \textit{if } A \textit{ is unbounded.} \end{cases}$$

Proposition 10 (The capacity of a set when $p = d$)**.** *Let $p = d \geq 2$. Then, for every set $A \subset \mathbb{R}^d$ we have:*

$$\mathrm{cap}_p(A) = 0 \qquad \textit{or} \qquad \mathrm{cap}_p(A) = +\infty.$$

Proof. For every integer $n \geq 1$, consider the ball B_n of radius n and center the origin. Since $\mathrm{cap}_d(B_n) = 0$ we can select a function $\phi_n \in W^{1,d}(\mathbb{R}^d)$ such that

$$0 \leq \phi_n \leq 1 \quad \text{on} \quad \mathbb{R}^d,$$
$$\phi_n = 1 \quad \text{on} \quad B_n,$$

and

$$\int_{\mathbb{R}^d} |\nabla\phi_n|^d \, dx \leq \frac{1}{n}.$$

Let now $A \subset \mathbb{R}^d$ be any set. If

$$\mathrm{cap}_d(A) < +\infty,$$

then there is a function $u \in W^{1,d}(\mathbb{R}^d)$ such that

$$0 \le u \le 1 \quad \text{on} \quad \mathbb{R}^d,$$

and

$$u = 1 \quad \text{in a neighborhood of} \quad A.$$

Consider the sequence of functions

$$u_n := u \vee \phi_n.$$

Then, $u_n \in W^{1,d}(\mathbb{R}^d)$ and

$$|\nabla u_n|^d = |\nabla(u \vee \phi_n)|^d = 0 \quad \text{on} \quad B_n,$$

$$|\nabla u_n|^d = |\nabla(u \vee \phi_n)|^d \le |\nabla u|^d + |\nabla \phi_n|^d \quad \text{on} \quad \mathbb{R}^d \setminus B_n.$$

Thus,

$$\int_{\mathbb{R}^d} |\nabla u_n|^d \, dx \le \int_{\mathbb{R}^d} |\nabla \phi_n|^d \, dx + \int_{\mathbb{R}^d \setminus B_n} |\nabla u|^d \, dx \le \frac{1}{n} + \int_{\mathbb{R}^d \setminus B_n} |\nabla u|^d \, dx.$$

Now since $u_n = 1$ in a neighborhood of A we get

$$\mathrm{cap}_d(A) \le \frac{1}{n} + \int_{\mathbb{R}^d \setminus B_n} |\nabla u|^d \, dx.$$

Since n is arbitrary, passing to the limit as $n \to +\infty$, we get that $\mathrm{cap}_d(A) = 0$. □

The case $p < d$. An alternative definition of the p-capacity

In this section we will focus on the case $p < d$. We will prove the following proposition.

Proposition 11 (An equivalent definition of the p-capacity)**.** *Let $d \ge 2$ and $p \in (1, d)$. For any set $A \subset \mathbb{R}^d$, the p-capacity of A is given by*

$$\mathrm{cap}_p(A) = \inf\Big\{ \int_{\mathbb{R}^d} |\nabla u|^p \, dx \ : \ u \in \dot{W}^{1,p}(\mathbb{R}^d), \ 0 \le u \le 1 \ \textit{on} \ \mathbb{R}^d, \ u = 1 \ \textit{a.e. in a neighborhood of} \ A \Big\},$$

where by $\dot{W}^{1,p}(\mathbb{R}^d)$ we denote the closure of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|u\|_{\dot{W}^{1,p}(\mathbb{R}^d)} = \|\nabla u\|_{L^p(\mathbb{R}^d)} + \|u\|_{L^{p_*}(\mathbb{R}^d)},$$

where as usual

$$p_* := \frac{pd}{d-p}.$$

Remark 12. *We notice that the Gagliardo-Nirenberg-Sobolev inequality guarantees that every function $u \in W^{1,p}(\mathbb{R}^d)$ lies in $L^{p_*}(\mathbb{R}^d)$. Thus, $W^{1,p}(\mathbb{R}^d)$ is contained in $\dot{W}^{1,p}(\mathbb{R}^d)$ and so, we immediately obtain the inequality*

$$\mathrm{cap}_p(A) \ge \inf\Big\{ \int_{\mathbb{R}^d} |\nabla u|^p \, dx \ : \ u \in \dot{W}^{1,p}(\mathbb{R}^d), \ 0 \le u \le 1 \ \textit{on} \ \mathbb{R}^d, \ u = 1 \ \textit{a.e. in a neighborhood of} \ A \Big\}.$$

Thus, in order to prove Proposition 11, we only need to show that the following inequality does hold:

$$\mathrm{cap}_p(A) \le \inf\Big\{ \int_{\mathbb{R}^d} |\nabla u|^p \, dx \ : \ u \in \dot{W}^{1,p}(\mathbb{R}^d), \ 0 \le u \le 1 \ \textit{on} \ \mathbb{R}^d, \ u = 1 \ \textit{a.e. in a neighborhood of} \ A \Big\}. \tag{1}$$

In order to prove the inequality (1), we will take a function $u \in \dot{W}^{1,p}(\mathbb{R}^d)$ and we will approximate it with functions in $W^{1,p}(\mathbb{R}^d)$. There are two different ways to do this: via "horizontal" truncations and via "vertical" truncations. The horizontal truncations are simply obtained by multiplying u with standard cut-off functions; they are easy to handle but they only work when the set A is bounded. The vertical truncations on the other hand are obtained by cutting u along its level sets; this argument works also for unbounded sets A. For the sake of completeness we present here both strategies.

Proof of Proposition 11 for bounded sets. As observed above we only need to prove the inequality (**??**). Consider a function $u \in \dot W^{1,p}(\mathbb{R}^d)$. For every $R > 0$ consider the function $\varphi_R(x) = \varphi(x/R)$, where $\varphi \in C_c^\infty(B_2)$ is a fixed function such that $\varphi = 1$ on B_1. Then, φ_R is supported in B_{2R} and equals 1 on B_R. We next compute

$$\begin{aligned}
\|\nabla u - \nabla(u\varphi_R)\|_{L^p(\mathbb{R}^d)} &= \|(1-\varphi_R)\nabla u + u\nabla\varphi_R\|_{L^p(\mathbb{R}^d)}\\
&\le \|(1-\varphi_R)\nabla u\|_{L^p(\mathbb{R}^d)} + \|u\nabla\varphi_R\|_{L^p(\mathbb{R}^d)}\\
&\le \|(1-\varphi_R)\nabla u\|_{L^p(\mathbb{R}^d)} + \|\nabla\varphi_R\|_{L^\infty(\mathbb{R}^d)}\|u\|_{L^p(\mathbb{R}^d)}\\
&= \|(1-\varphi_R)\nabla u\|_{L^p(\mathbb{R}^d)} + \frac{\|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)}}{R}\|u\|_{L^p(\mathbb{R}^d)}\\
&\le \|\nabla u\|_{L^p(\mathbb{R}^d\setminus B_R)} + \frac{\|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)}}{R}\|u\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

For $R \to +\infty$, we have that

$$\|\nabla u - \nabla(u\varphi_R)\|_{L^p(\mathbb{R}^d)} \to 0,$$

so

$$\int_{\mathbb{R}^d} |\nabla u|^p\,dx = \lim_{R\to+\infty}\int_{\mathbb{R}^d} |\nabla(u\varphi_R)|^p\,dx.$$

This implies that

$$\begin{aligned}
&\inf\Big\{\int_{\mathbb{R}^d}|\nabla u|^p\,dx \ :\ u\in \dot W^{1,p}(\mathbb{R}^d),\ 0\le u\le 1 \ \text{ on }\ \mathbb{R}^d,\ u=1\ \text{ a.e. in a neighborhood of } A\Big\}\\
&\quad = \inf\Big\{\int_{\mathbb{R}^d}|\nabla u|^p\,dx \ :\ u\in \dot W^{1,p}(\mathbb{R}^d),\ 0\le u\le 1 \ \text{ on }\ \mathbb{R}^d,\ u=1\ \text{ a.e. in a neighborhood of } A,\\
&\qquad\qquad\qquad u = 0 \text{ outside some ball } B_R \text{ containing } A\Big\}.
\end{aligned}$$

But now, since $p_* > p$, any function $u \in \dot W^{1,p}(\mathbb{R}^d)$, which is constantly zero outside some ball B_R, lies in $W^{1,p}(\mathbb{R}^d)$. Thus, we get the opposite inequality

$$\mathrm{cap}_p(A) \le \inf\Big\{\int_{\mathbb{R}^d}|\nabla u|^p\,dx \ :\ u\in \dot W^{1,p}(\mathbb{R}^d),\ 0\le u\le 1 \ \text{ on }\ \mathbb{R}^d,\ u=1\ \text{ a.e. in a neighborhood of } A\Big\},$$

which concludes the proof. □

Proof of Proposition 11 for general sets. Thanks to Remark 12, we only need to prove the inequality (1). Consider a function $u \in \dot W^{1,p}(\mathbb{R}^d)$ such that

$$0\le u\le 1 \ \text{ on }\ \mathbb{R}^d \qquad \text{and} \qquad u=1\ \text{ a.e. in a neighborhood of } A.$$

For every $t \in (0,1)$, we define

$$u_t(x) = \frac{1}{1-t}(u(x)-t)_+.$$

Then

$$0 \le u_t \le 1 \quad \text{on} \quad \mathbb{R}^d,$$

and

$$u_t = 1 \quad \text{on the set} \quad \{u=1\}.$$

Moreover, since

$$\{u_t > 0\} = \{u > t\},$$

and

$$|\{u>t\}| \le \frac{1}{t^{p_*}}\int_{\mathbb{R}^d} u^{p_*}\,dx < +\infty,$$

we get that

$$u_t \in L^p(\mathbb{R}^d),$$

so that $u_t \in W^{1,p}(\mathbb{R}^d)$. Finally, we have

$$\int_{\mathbb{R}^d}|\nabla u_t|^p\,dx = \frac{1}{(1-t)^p}\int_{\{u>t\}}|\nabla u|^p\,dx,$$

so taking the limit as $t \to 1^-$, we get

$$\lim_{t\to+\infty}\int_{\mathbb{R}^d}|\nabla u_t|^p\,dx = \int_{\{u>0\}}|\nabla u|^p\,dx = \int_{\mathbb{R}^d}|\nabla u|^p\,dx.$$

This gives the inequality

$$\int_{\mathbb{R}^d} |\nabla u|^p\,dx \le \inf\left\{\int_{\mathbb{R}^d} |\nabla v|^p\,dx \ :\ v\in \dot W^{1,p}(\mathbb{R}^d),\ 0\le v\le 1 \text{ on } \mathbb{R}^d,\ v=1 \text{ a.e. in a neighborhood of } A\right\},$$

which after taking the infimum over u, concludes the proof of (1) and of Proposition 11 for general sets $A\subset\mathbb{R}^d$. □

The case $p<d$. Measure and capacity

Proposition 13. *For every $d\ge 2$ and $p\in[1,d)$, there is a constant $C=C(d,p)>0$ such that*

$$|A|^{\frac{d-p}{d}} \le C(d,p)\,\mathrm{cap}_p(A)\quad \text{for every set}\quad A\subset\mathbb{R}^d,$$

where

$$p_* := \frac{pd}{d-p}.$$

Proof. Thanks to the Gagliardo-Nirenberg-Sobolev inequality, there is a constant $C=C(d,p)>0$ such that

$$\left(\int_{\mathbb{R}^d} |u|^{p_*}\,dx\right)^{\frac{d-p}{d}} \le C\int_{\mathbb{R}^d} |\nabla u|^p\,dx,$$

for every $u\in W^{1,p}(\mathbb{R}^d)$. In particular, if $u\ge 1$ almost-everywhere on A, then

$$|A|^{\frac{d-p}{d}} \le C\int_{\mathbb{R}^d} |\nabla u|^p\,dx.$$

Taking the infimum over all $u\in W^{1,p}(\mathbb{R}^d)$ such that $u\ge 1$ on a neighborhood of A, we get the claim. □

The case $p<d$. The p-capacity along monotone sequences

In this section we will prove the following proposition

Proposition 14. *Let $d\ge 2$ and $p\in(1,d)$. Let $(A_n)_{n\ge 1}$ be an increasing sequence of sets in $\mathbb{R}^d$ and let*

$$A=\bigcup_{n=1}^{+\infty} A_n.$$

Then,

$$\mathrm{cap}_p(A) = \lim_{n\to+\infty}\mathrm{cap}_p(A_n) = \sup_{n\to+\infty}\mathrm{cap}_p(A_n).$$

Proof. Since A_n is an increasing sequence the equality

$$\lim_{n\to+\infty}\mathrm{cap}_p(A_n) = \sup_{n\ge 1}\left\{\mathrm{cap}_p(A_n)\right\},$$

holds true. Moreover, since A contains A_n for every n, we have

$$\mathrm{cap}_p(A)\ge \mathrm{cap}_p(A_n)\quad \text{for all}\quad n\ge 1,$$

so we get

$$\mathrm{cap}_p(A)\ge \lim_{n\to+\infty}\mathrm{cap}_p(A_n).$$

In particular, we only need to prove the opposite inequality

$$\mathrm{cap}_p(A)\le \lim_{n\to+\infty}\mathrm{cap}_p(A_n).$$

in the case

$$\sup_{n\ge 1}\left\{\mathrm{cap}_p(A_n)\right\} < +\infty.$$

In what follows we fix some arbitrary $\varepsilon>0$. We aim to show that

$$\mathrm{cap}_p(A)\le \lim_{n\to+\infty}\mathrm{cap}_p(A_n)+\varepsilon.$$

Since, for each n, the set A_n has finite capacity, we can find a function u_n such that:

- $u_n \in W^{1,p}(\mathbb{R}^d)$;
- $0 \le u_n \le 1$ on $\mathbb{R}^d$;
- $u_n = 1$ on an open set Ω_n containing A_n;

and

$$\mathrm{cap}_p(A_n) \le \int_{\mathbb{R}^d} |\nabla u_n|^p \, dx \le \mathrm{cap}_p(A_n) + \frac{\varepsilon}{2^n}. \tag{2}$$

We next define the sequence of functions

$$h_n := u_1 \vee u_2 \vee \cdots \vee u_n.$$

Since the functions h_n can be obtained inductively as

$$h_1 = u_1 \quad \text{and} \quad h_n = h_{n-1} \vee u_n \quad \text{for all} \quad n \ge 2,$$

we get that, for all $n \ge 2$,

- $h_n \in W^{1,p}(\mathbb{R}^d)$;
- $0 \le h_n \le 1$ on $\mathbb{R}^d$;
- $h_{n-1} \le h_n$;
- $h_n = 1$ on the union of open sets $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n$.

Moreover, since we have the formula

$$\int_{\mathbb{R}^d} |\nabla(u_n \vee h_{n-1})|^p \, dx + \int_{\mathbb{R}^d} |\nabla(u_n \wedge h_{n-1})|^p \, dx = \int_{\mathbb{R}^d} |\nabla u_n|^p \, dx + \int_{\mathbb{R}^d} |\nabla h_{n-1}|^p \, dx,$$

and since

$$u_n \wedge h_{n-1} \ge 1 \quad \text{on the open set} \quad \Omega_n \cap \left(\bigcup_{k=1}^{n-1} \Omega_k \right),$$

which contains A_{n-1}, we get that

$$\int_{\mathbb{R}^d} |\nabla(u_n \vee h_{n-1})|^p \, dx + \mathrm{cap}_p(A_{n-1}) = \int_{\mathbb{R}^d} |\nabla u_n|^p \, dx + \int_{\mathbb{R}^d} |\nabla h_{n-1}|^p \, dx,$$

which combined with the inequality (2) gives

$$\int_{\mathbb{R}^d} |\nabla h_n|^p \, dx + \mathrm{cap}_p(A_{n-1}) \le \mathrm{cap}_p(A_n) + \frac{\varepsilon}{2^n} + \int_{\mathbb{R}^d} |\nabla h_{n-1}|^p \, dx,$$

for every $n \ge 2$. Now, summing up these inequalities, together with

$$\int_{\mathbb{R}^d} |\nabla h_1|^p \, dx \le \mathrm{cap}_p(A_1) + \frac{\varepsilon}{2},$$

we get that

$$\int_{\mathbb{R}^d} |\nabla h_n|^p \, dx \le \mathrm{cap}_p(A_n) + \varepsilon \sum_{k=1}^{n} 2^{-k} \le \mathrm{cap}_p(A_n) + \varepsilon\,.$$

In particular, the sequence h_n is monotone increasing and bounded in $L^{p_*}(\mathbb{R}^d)$. By the monotone convergence theorem, h_n converges to the function

$$h(x) := \sup_{n \ge 1} h_n(x),$$

strongly in $L^{p_*}(\mathbb{R}^d)$. By construction, we have that

- $0 \le h \le 1$ on $\mathbb{R}^d$;
- $h = 1$ on the union of the open set $\Omega := \bigcup_{n \ge 1} \Omega_n$, which clearly contains A.

Moreover, since we have a uniform bound on the L^p norm of the gradients

$$\int_{\mathbb{R}^d} |\nabla h_n|^p \, dx \le \varepsilon + \sup_{n \ge 1} \left\{ \mathrm{cap}_p(A_n) \right\},$$

we have that $h \in \dot{W}^{1,p}(\mathbb{R}^d)$ and that ∇h_n converges weakly in L^p to ∇h. By the semicontinuity of the norm of the gradient with respect to the weak convergence, we get

$$\int_{\mathbb{R}^d} |\nabla h|^p \, dx \le \liminf_{n \to +\infty} \int_{\mathbb{R}^d} |\nabla h_n|^p \, dx \le \varepsilon + \sup_{n \ge 1} \left\{ \mathrm{cap}_p(A_n) \right\}.$$

Now, since the p-capacity is given by the formula

$$\operatorname{cap}_p(A) = \inf\left\{ \int_{\mathbb{R}^d} |\nabla v|^p \, dx \ : \ v \in \dot{W}^{1,p}(\mathbb{R}^d), \ 0 \le v \le 1 \ \text{ on } \ \mathbb{R}^d, \ v = 1 \ \text{a.e. in a neighborhood of } A \right\},$$

we get that

$$\operatorname{cap}_p(A) \le \varepsilon + \sup_{n\ge 1} \left\{\operatorname{cap}_p(A_n)\right\},$$

which concludes the proof. □

Remark 15. *We notice that Proposition 14 is false when $p \ge d$. Indeed, if we pick any set $A \subset \mathbb{R}^d$ such that $\operatorname{cap}_p(A) = +\infty$ (for instance, one can simply choose $A = \mathbb{R}^d$), then*

$$\lim_{n\to+\infty} \operatorname{cap}_p(A \cap B_n) = 0 \ne +\infty = \operatorname{cap}_p(A).$$

The case $p < d$. Outer measure property

From this proposition, we immediately obtain

Proposition 16 (σ-subadditivity of cap_p)**.** *Let $p \in [1, +\infty)$ and $d \ge 1$. Suppose that $(A_n)_{n\ge 1}$ is a sequence of sets in $\mathbb{R}^d$ and let $A = \cup_n A_n$. Then,*

$$\operatorname{cap}_p(A) \le \sum_{n=1}^{+\infty} \operatorname{cap}_p(A_n).$$

Proof of Proposition 16. Consider the sequence of sets

$$\widetilde{A}_n := \bigcup_{k=1}^{n} A_k.$$

Thanks to the subadditivity of the capacity we have

$$\operatorname{cap}_p(\widetilde{A}_n) \le \sum_{k=1}^{n} \operatorname{cap}_p(A_k).$$

Now, since $\widetilde{A}_n$ is an increasing sequence and since

$$A = \bigcup_{n=1}^{\infty} \widetilde{A}_n,$$

so thanks to Proposition 14, we can take the limit on the left-hand side obtaining

$$\operatorname{cap}_p(A) = \lim_{n\to+\infty} \operatorname{cap}_p(\widetilde{A}_n) \le \sum_{n=1}^{+\infty} \operatorname{cap}_p(A_n).$$

□

Remark 17. *We notice that also Proposition 16 is false when $p \ge d$. As in Remark 15, we take any set A for which there are no functions $u \in W^{1,p}(\mathbb{R}^d)$ such that $u = 1$ on A (for instance, one may take A to be a set of infinite Lebesgue measure). Then, by the definition of the p-capacity,*

$$\operatorname{cap}_p(A) = +\infty.$$

On the other hand, for every ball B_n of radius n, we have

$$\operatorname{cap}_p(A \cap B_n) \le \operatorname{cap}_p(B_n) = 0.$$

Thus, taking $A_n = A \cap B_n$, we get

$$\sum_{n=1}^{+\infty} \operatorname{cap}_p(A_n) = 0 \ne +\infty = \operatorname{cap}_p(A).$$

Scaling, translations and rotations

In this section we show how the capacity of a set A behaves with respect to isometries (translations and rotations) and with respect to homotheties. Both properties are an immediate consequence of the definition of the p-capacity. We give the detailed proofs for the sake of completeness.

Lemma 18. *Let $d \geq 2$ and $p \in (1, +\infty)$. Then, for any set $A \subset \mathbb{R}^d$ and any real number $t > 0$, we have*
$$\operatorname{cap}_p(tA) = t^{d-p} \operatorname{cap}_p(A).$$

Proof. We will show that
$$\operatorname{cap}_p(tA) \leq t^{d-p} \operatorname{cap}_p(A). \tag{3}$$
Indeed, suppose that $\operatorname{cap}_p(A) < +\infty$. Then, for any $\varepsilon > 0$, there is a function $u \in W^{1,p}(\mathbb{R}^d)$ such that
$$0 \leq u \leq 1 \quad \text{on} \quad \mathbb{R}^d,$$
$u = 1$ on an open set Ω containing A, and
$$\operatorname{cap}_p(A) \leq \int_{\mathbb{R}^d} |\nabla u|^p \, dx \leq \operatorname{cap}_p(A) + \varepsilon.$$
Consider the function
$$u_t(x) := u(x/t).$$
Then, $0 \leq u_t \leq 1$ on $\mathbb{R}^d$, $u_t = 1$ on the open set $t\Omega$ (which clearly contains tA), and
$$\int_{\mathbb{R}^d} |\nabla u_t(x)|^p \, dx = \int_{\mathbb{R}^d} \left| \frac{1}{t} \nabla u(x/t) \right|^p \, dx = t^{d-p} \int_{\mathbb{R}^d} |\nabla u|^p \,(y) \, dy \leq t^{d-p} \Big(\operatorname{cap}_p(A) + \varepsilon \Big).$$
Since ε is arbitrary, we get precisely (3).

Finally, applying (3) to tA and $1/t$, we get
$$\operatorname{cap}_p(A) = \operatorname{cap}_p\left(\frac{1}{t}(tA) \right) \leq \left(\frac{1}{t} \right)^{d-p} \operatorname{cap}_p(tA),$$
which gives the opposite inequality
$$t^{d-p} \operatorname{cap}_p(A) \leq \operatorname{cap}_p(tA),$$
and concludes the proof. □

Lemma 19. *Let $d \geq 2$ and $p \in (1, +\infty)$. Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be an isometric one-to-one map from $\mathbb{R}^d$ into itself. Then, for any set $A \subset \mathbb{R}^d$ we have*
$$\operatorname{cap}_p(\Phi(A)) = \operatorname{cap}_p(A).$$

Proof. Since the inverse of Φ is also an isometry, it is sufficient to show that
$$\operatorname{cap}_p(\Phi(A)) \leq \operatorname{cap}_p(A). \tag{4}$$
Indeed, suppose that $\operatorname{cap}_p(A) < +\infty$. Then, for any $\varepsilon > 0$, there is a function $u \in W^{1,p}(\mathbb{R}^d)$ such that
$$0 \leq u \leq 1 \quad \text{on} \quad \mathbb{R}^d,$$
$u = 1$ on an open set Ω containing A, and
$$\operatorname{cap}_p(A) \leq \int_{\mathbb{R}^d} |\nabla u|^p \, dx \leq \operatorname{cap}_p(A) + \varepsilon.$$
Consider the function
$$w(x) := u(\Phi^{-1}(x)).$$
Then, $0 \leq w \leq 1$ on $\mathbb{R}^d$, and $w = 1$ on the open set $\Phi(\Omega)$ (which clearly contains $\Phi(A)$). Moreover, since for each $x \in \mathbb{R}^d$, the linear map $J\Phi(x) : \mathbb{R}^d \to \mathbb{R}^d$ is also an isometry, we get that
$$\int_{\mathbb{R}^d} |\nabla w(x)|^p \, dx = \int_{\mathbb{R}^d} \left| J\Phi^{-1}(x)[\nabla u(\Phi^{-1}(x))] \right|^p \, dx = \int_{\mathbb{R}^d} \left| \nabla u(\Phi^{-1}(x)) \right|^p |\det J\Phi^{-1}(x)| \, dx = \int_{\mathbb{R}^d} |\nabla u(x)|^p \, dx.$$
Since w is an admissible competitor in the definition of the p-capacity of $\Phi(A)$, we get that
$$\operatorname{cap}_p(\Phi(A)) \leq \int_{\mathbb{R}^d} |\nabla w|^p \, dx = \int_{\mathbb{R}^d} |\nabla u(x)|^p \, dx \leq \operatorname{cap}_p(A) + \varepsilon.$$
Finally, since ε is arbitrary, we get precisely (4). □

The case $p < d$. The capacity of a point in $\mathbb{R}^d$, $d \geq 2$.

Proposition 20. *Let $d \geq 2$ and $p \in (1,d)$. Then,*

$$\mathrm{cap}_p(\{x_0\}) = 0 \quad \textit{for every point} \quad x_0 \in \mathbb{R}^d.$$

Proof. Thanks to the translation invariance of the capacity, it is sufficient to prove the claim in the case $x_0 = 0$. Using the monotonicity and the scaling of the capacity we have

$$\mathrm{cap}_p(\{0\}) \leq \mathrm{cap}_p(B_r) = r^{d-p}\,\mathrm{cap}_p(B_1) \quad \text{for all} \quad r > 0.$$

Taking the limit as $r \to 0$, we get the claim. □

Corollary 21. *Let $d \geq 2$ and $p \in (1,d)$. Then,*

$$\mathrm{cap}_p(A) = 0,$$

for every set A composed of at most countably many points.

Proof. Suppose that $A = \{x_n\}_{n\geq 1}$. Then, thanks to Proposition 16 and Proposition 20, we get

$$\mathrm{cap}_p(A) \leq \sum_{n=1}^{+\infty} \mathrm{cap}_p(\{x_n\}) = 0.$$

□