Relative p-capacity

This section is dedicated to the p-capacity of a bounded set in $A \subset \mathbb{R}^d$ with respect to an open set Ω containing $\overline{A}$; we notice that this notion of *relative capacity* (usually indicated by $\mathrm{cap}_p(A,\Omega)$) can be defined for every $p \geq 1$. We are mainly interested in the case $p \in (1,d]$ in which the relative capacity allows to define a non-trivial family of sets of zero capacity in $\mathbb{R}^d$. Precisely, we will say that a set $A \subset \mathbb{R}^d$ has null capacity if

$$\mathrm{cap}_p\big(A \cap B_r(x); B_{2r}(x)\big) = 0 \quad \text{for all balls} \quad B_r(x) \subset \mathbb{R}^d. \tag{1}$$

We notice that the relative capacity will turn out to be a more suitable tool for defining sets of null capacity. Indeed, we will show that in the subcritical case $p < d$, we have

$$\text{“}A \text{ has null capacity”} \quad \Leftrightarrow \quad \mathrm{cap}_p(A) = 0,$$

so this definition is consistent with the classical definition of sets of null capacity in $\mathbb{R}^d$, which is discussed in detail in the book by Evans and Gariepy. On the other hand, in the critical case $p = d \geq 2$, all bounded sets $A \subset \mathbb{R}^d$ are such that $\mathrm{cap}_d(A) = 0$, while definition (1) produces a non-trivial class of sets of null capacity and a theory that allows to treat all cases $p \in (1,d]$ at once.

DEFINITION

We recall that the following notation:

given a set $A \subset \mathbb{R}^d$ and an open set $\Omega \subset \mathbb{R}^d$,
we say that $A \Subset \Omega$ if the closure $\overline{A}$ is a compact set contained in Ω.

The definition of relative p-capacity is the following.

Definition 1 (Relative p-capacity)**.** *Let $d \geq 2$, $p \in [1,+\infty)$, and let Ω be a bounded open set in $\mathbb{R}^d$. For every set $A \Subset \Omega$, we define the p-capacity of A relative to Ω as*

$$\mathrm{cap}_p(A;\Omega) = \inf\Big\{ \int_\Omega |\nabla u|^p\,dx \ :\ u \in W^{1,p}_0(\Omega),\ u \geq 1 \quad \textit{almost-everywhere in a neighborhood of } A\Big\}.$$

Proposition 2 (Equivalent definitions of the relative p-capacity)**.** *Let $p \in [1,+\infty)$ and $d \geq 2$.*

(i) *The p-capacity of any set $A \Subset \Omega$ can be computed as follows*

$$\mathrm{cap}_p(A;\Omega) = \inf\Big\{ \int_\Omega |\nabla u|^p\,dx \ :\ u \in W^{1,p}_0(\Omega),\ 0 \leq u \leq 1 \ \textit{on}\ \Omega,\ u = 1\ \textit{a.e. in a neighborhood of } A\Big\};$$

(ii) *For any open set A such that $A \Subset \Omega$ we have*

$$\mathrm{cap}_p(A;\Omega) = \inf\Big\{ \int_\Omega |\nabla u|^p\,dx \ :\ u \in W^{1,p}_0(\Omega),\ 0 \leq u \leq 1\ \textit{on}\ \mathbb{R}^d,\ u = 1\ \textit{on}\ A\Big\}.$$

Proof. For every $u \in W^{1,p}_0(\Omega)$, we have that $0 \vee u \wedge 1 \in W^{1,p}_0(\Omega)$ and that its weak gradient is given by

$$\nabla(0 \vee u \wedge 1) = \chi_{\{0<u<1\}}\nabla u,$$

so that

$$\int_\Omega |\nabla(0 \vee u \wedge 1)|^p\,dx = \int_{\{0<u<1\}} |\nabla u|^p\,dx \leq \int_\Omega |\nabla u|^p\,dx.$$

This gives immediately (i); (ii) follows from (i) since any open set A is an open neighborhood of itself. ☐

MONOTONICITY WITH RESPECT TO THE INCLUSION

Proposition 3 (Monotonicity with respect to the inclusion)**.** *Let $p \in [1,+\infty)$ and $d \geq 2$. Let Ω be a bounded open set in $\mathbb{R}^d$. Suppose that $A_1 \Subset \Omega$ and $A_2 \Subset \Omega$. If $A_1 \subset A_2$, then $\mathrm{cap}_p(A_1;\Omega) \leq \mathrm{cap}_p(A_2;\Omega)$.*

Proof. This follows immediately from the definition of $\mathrm{cap}_p(A_1;\Omega)$ and $\mathrm{cap}_p(A_2;\Omega)$. ☐

Subadditivity of the relative capacity

Proposition 4 (Subadditivity of the relative capacity)**.** *Let $p \in [1,+\infty)$ and $d \geq 2$. Let Ω be a bounded open set in $\mathbb{R}^d$. Then, for all sets $A_1 \Subset \Omega$ and $A_2 \Subset \Omega$, we have*

$$\mathrm{cap}_p(A_1 \cup A_2;\Omega) \leq \mathrm{cap}_p(A_1;\Omega) + \mathrm{cap}_p(A_2;\Omega).$$

Proof. Consider two functions $u_1, u_2 \in W^{1,p}_0(\Omega)$ such that, for $j = 1,2$,

$$u_j \geq 1 \quad \text{in a neighborhood of} \quad A_j.$$

Then, $u_1 \vee u_2 \in W^{1,p}_0(\Omega)$ and

$$u_1 \vee u_2 \geq 1 \quad \text{in a neighborhood of} \quad A_1 \cup A_2.$$

Moreover, we have the estimate

$$\mathrm{cap}_p(A_1 \cup A_2;\Omega) \leq \int_\Omega |\nabla(u_1 \vee u_2)|^p\,dx \leq \int_\Omega |\nabla(u_1 \vee u_2)|^p\,dx + \int_\Omega |\nabla(u_1 \wedge u_2)|^p\,dx = \int_\Omega |\nabla u_1|^p\,dx + \int_\Omega |\nabla u_2|^p\,dx.$$

Taking the infimum over all functions $u_1, u_2 \in W^{1,p}_0(\Omega)$ satisfying (**??**), we get the claim. ☐

Capacitary functions on open sets

When $A \Subset \Omega$ are bounded open subsets of $\mathbb{R}^d$ and $p \in (1,+\infty)$, the relative p-capacity of A with respect to Ω is realized by a (unique) *capacitary function* $u_{A,\Omega} \in W^{1,p}_0(\Omega)$ which depends on A, Ω and p, and is constantly equal to 1 on the set A. We discuss the existence and the properties of $u_{A,\Omega}$ in the

Proposition 5 (Existence and uniqueness)**.** *Let $d \geq 2$ and $p \in (1,+\infty)$. Suppose that Ω is a bounded open set of $\mathbb{R}^d$ and that $A \Subset \Omega$ is an open set. Then, there is a function $u_{A,\Omega} \in W^{1,p}_0(\Omega)$ that solves the variational problem*

$$\min\Big\{ \int_\Omega |\nabla u|^p\,dx \ :\ u \in W^{1,p}_0(\Omega),\ u \geq 1 \ \ a.e.\ \ su \ \ A \Big\}.$$

Moreover,

- *the minimizer $u_{A,\Omega}$ is unique;*
- $0 \leq u_{A,\Omega} \leq 1$ *on Ω;*
- $\mathrm{cap}(A;\Omega) = \displaystyle\int_\Omega |\nabla u_{A,\Omega}|^p\,dx.$

Proposition 6 (Monotonicity with respect to A)**.** *Let $d \geq 2$ and $p \in (1,+\infty)$. Suppose that Ω is a bounded open set of $\mathbb{R}^d$ and that $A_1 \Subset \Omega$ and $A_2 \Subset \Omega$ are open sets such that $A_1 \subset A_2$. Then,*

$$u_{A_1,\Omega} \leq u_{A_2,\Omega} \qquad \textit{and} \qquad \int_\Omega |\nabla u_{A_2,\Omega}|^p\,dx \geq \int_\Omega |\nabla u_{A_1,\Omega}|^p\,dx.$$

Proposition 7 (Monotonicity with respect to Ω)**.** *Let $d \geq 2$ and $p \in (1,+\infty)$. Suppose that $\Omega_1 \subset \Omega_2$ are two bounded open set of $\mathbb{R}^d$ and that $A \Subset \Omega_1$ is an open set. Then,*

$$u_{A,\Omega_1} \leq u_{A,\Omega_2} \qquad \textit{and} \qquad \int_{\mathbb{R}^d} |\nabla u_{A,\Omega_1}|^p\,dx \geq \int_{\mathbb{R}^d} |\nabla u_{A,\Omega_2}|^p\,dx.$$

Smooth deformations and capacity

Let Ω be an open set in $\mathbb{R}^d$. We recall that a map

$$\Phi = (\Phi_1, \dots, \Phi_d) : \Omega \to \mathbb{R}^d,$$

is differentiable at a point $x \in \Omega$ if

$$\Phi(x+h) = \Phi(x) + J\Phi(x)[h] + o(|h|),$$

where $J\Phi$ is the Jacobian

$$J\Phi(x) = \begin{pmatrix} \partial_1\Phi_1(x) & \dots & \partial_1\Phi_d(x) \\ \vdots & \ddots & \vdots \\ \partial_d\Phi_1(x) & \dots & \partial_d\Phi_d(x) \end{pmatrix}.$$

Moreover, we will say that Φ is of class C^1 if Φ is differentiable at every point $x \in \Omega$ and if the functions

$$\partial_i \Phi_j : \Omega \to \mathbb{R}$$

are continuous for all $i = 1, \dots, d$ and $j = 1, \dots, d$.

Finally, we recall that for any $d \times d$ matrix $A = (a_{ij})_{ij}$, we denote by $\|A\|_2$ the norm

$$\|A\|_2 := \left(\sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}^2 \right)^{1/2}.$$

Proposition 8. *Let $d \geq 2$ and $p \in [1, +\infty)$. Let Ω and $\widetilde{\Omega}$ be bounded open sets in $\mathbb{R}^d$ and let*

$$\Phi : \Omega \to \widetilde{\Omega}$$

be a C^1 regular map with inverse $\Psi : \widetilde{\Omega} \to \Omega$, which is also of class C^1 on $\widetilde{\Omega}$. Suppose, moreover, that there is a constant $L < +\infty$ such that

$$\sup_{x \in \Omega} \|J\Phi(x)\|_2 + \sup_{y \in \widetilde{\Omega}} \|J\Psi(y)\|_2 \leq L.$$

Then, for every set $A \Subset \Omega$, we have that $\widetilde{A} := \Phi(A) \Subset \widetilde{\Omega}$ and

$$\mathrm{cap}_p(\widetilde{A}; \widetilde{\Omega}) \leq L^{d+2}\, \mathrm{cap}_p(A; \Omega).$$

Proof. Let $u \in W_0^{1,p}(\Omega)$ be such that $0 \leq u \leq 1$ on Ω and $u \equiv 1$ on an open set U such that

$$A \subset U \subset \Omega.$$

Consider the function

$$\widetilde{u} : \widetilde{\Omega} \to \mathbb{R}\ , \qquad \widetilde{u}(y) = u(\Psi(y)).$$

It is clear that $0 \leq \widetilde{u} \leq 1$ on $\widetilde{\Omega}$ and that

$$\widetilde{u} = 1 \quad \text{on} \quad \widetilde{U},$$

where $\widetilde{U} := \Phi(U)$ is an open set containing $\widetilde{A}$. Moreover, we know that $\widetilde{u} \in W_0^{1,p}(\widetilde{\Omega})$ and that

$$\begin{aligned}
\int_{\widetilde{\Omega}} |\nabla \widetilde{u}(y)|^p \, dy &= \int_{\widetilde{\Omega}} |\nabla (u(\Psi(y)))|^p \, dy \\
&= \int_{\widetilde{\Omega}} |J\Psi(y) \nabla u(\Psi(y))|^p \, dy \\
&= \int_{\Omega} |J\Psi(\Phi(x)) \nabla u(x)|^p \, |\det J\Phi(x)| \, dx \leq L^{d+2} \int_{\Omega} |\nabla u(x)|^p \, dx,
\end{aligned}$$

where we have used the change of variables $y = \Phi(x)$, $x = \Psi(y)$. Since, by definition

$$\mathrm{cap}_p(\widetilde{A}; \widetilde{\Omega}) \leq \int_{\widetilde{\Omega}} |\nabla \widetilde{u}(y)|^p \, dy,$$

we get that

$$\mathrm{cap}_p(\widetilde{A}; \widetilde{\Omega}) \leq L^{d+2} \int_{\Omega} |\nabla u(x)|^p \, dx.$$

Finally, taking the infimum with respect to u, we get the claim. ☐

SOME EXAMPLES IN $\mathbb{R}^2$

Proposition 9 (The capacity of a ball B_r in B_R)**.** *Let $p = d = 2$ and $0 < r < R < +\infty$.*
Let Ω be the ball of radius R in $\mathbb{R}^2$ and let A be the ball of radius r in $\mathbb{R}^2$.
Then, the relative 2-capacity of B_r with respect to B_R is given by

$$\mathrm{cap}_2(B_r; B_R) = \frac{2\pi}{\ln(R/r)}.$$

Proof. Consider the function

$$h(x) := \begin{cases} 1 & \text{when} \quad |x| \le r; \\ 0 & \text{when} \quad |x| \ge R; \\ \dfrac{\ln(|x|/R)}{\ln(r/R)} & \text{when} \quad r \le |x| \ge R. \end{cases}$$

Then, h is a Sobolev function, $h \in W_0^{1,2}(B_R)$, with weak gradient given by

$$\nabla h(x) = -\frac{1}{\ln(R/r)} \frac{x}{|x|^2} \chi_{B_R \setminus \overline{B}_r}(x).$$

Moreover, $\Delta h = 0$ in $B_R \setminus \overline{B}_r$ which can be written as

$$\int_{B_R \setminus \overline{B}_r} \nabla h \cdot \nabla \phi \, dx = 0 \quad \text{for all} \quad \phi \in C_c^\infty(B_R \setminus \overline{B}_r),$$

and by density, we get

$$\int_{B_R \setminus \overline{B}_r} \nabla h \cdot \nabla \phi \, dx = 0 \quad \text{for all} \quad \phi \in W_0^{1,2}(B_R \setminus \overline{B}_r).$$

Consider now a function $u \in W_0^{1,2}(B_R)$ such that $u = 1$ on B_r. Since $B_R \setminus \overline{B}_r$ is a C^1 domain and since $u - h \equiv 0$ on $\mathbb{R}^2 \setminus \Big(B_R \setminus \overline{B}_r\Big)$, we get that $u - h \in W_0^{1,2}(B_R \setminus \overline{B}_r)$, so we can use it in place of the test function ϕ above:

$$\int_{B_R \setminus \overline{B}_r} \nabla h \cdot \nabla(u - h) \, dx = 0.$$

Thus, writing u in the form $h + (u - h)$, we get the estimate

$$\int_{B_R} |\nabla u|^2 \, dx = \int_{B_R} |\nabla h|^2 \, dx + \int_{B_R} |\nabla(u - h)|^2 \, dx \ge \int_{B_R} |\nabla h|^2 \, dx,$$

which proves that

$$\mathrm{cap}_2(B_r; B_R) = \int_{B_R} |\nabla h|^2 \, dx.$$

Finally, computing explicitly this last integral we get

$$\int_{B_R} |\nabla h|^2 \, dx = 2\pi \int_r^R \frac{1}{(\ln(R/r))^2} \frac{1}{\rho^2} \rho \, d\rho = \frac{2\pi}{\ln(R/r)},$$

which concludes the proof. □

Proposition 10 (The capacity of a circumference in B_R)**.** *Let $p = d = 2$ and $0 < r < R < +\infty$. Let Ω be the ball of radius R in $\mathbb{R}^2$ and let A be the circumference of radius r in $\mathbb{R}^2$. Then, the relative 2-capacity of ∂B_r with respect to B_R is given by*

$$\mathrm{cap}_2(\partial B_r; B_R) = \frac{2\pi}{\ln(R/r)}.$$

Proof. Suppose that $u \in W_0^{1,2}(B_R)$ is a function such that $u = 1$ in an open neighborhood $\mathcal{U}$ of ∂B_r. Then, we can decompose u as

$$u = v + \phi,$$

where $\phi \in W_0^{1,2}(B_r)$ and $v \equiv 1$ on $B_r \cup \mathcal{U}$. In particular,

$$\int_{B_R} |\nabla u|^2 \, dx = \int_{B_R} |\nabla \varphi|^2 \, dx + \int_{B_R} |\nabla v|^2 \, dx \ge \int_{B_R} |\nabla v|^2 \, dx \ge \mathrm{cap}_2(\overline{B}_r; B_R).$$

Since u is arbitrary we get

$$\mathrm{cap}_2(\partial B_r; B_R) \ge \mathrm{cap}_2(\overline{B}_r; B_R).$$

On the other hand, by the monotonicity of the relative capacity we have

$$\mathrm{cap}_2(\partial B_r; B_R) \le \mathrm{cap}_2(\overline{B}_r; B_R),$$

so we get the equality

$$\mathrm{cap}_2(\partial B_r; B_R) = \mathrm{cap}_2(\overline{B}_r; B_R).$$

Finally, using again the monotonicity of the capacity we have that for all $\varepsilon > 0$

$$\frac{2\pi}{\ln(R/r)} = \mathrm{cap}_2(B_r; B_R) \le \mathrm{cap}_2(\overline{B}_r; B_R) \le \mathrm{cap}_2(B_{r+\varepsilon}; B_R) = \frac{2\pi}{\ln(R/(r+\varepsilon))},$$

so passing to the limit as $\varepsilon \to 0$, we get the claim. $\square$

Proposition 11 (The capacity of a segment in B_R)**.** *Let $p = d = 2$, $1 \le R \le +\infty$ and $0 < \ell < 1/8$. Let Ω be the ball of radius R in $\mathbb{R}^2$ and let Σ_ℓ be the segment*

$$\Sigma_\ell := \Big\{(x, 0) \ : \ -\ell \le x <\le \Big\}.$$

Then, the relative 2-capacity of Σ_ℓ with respect to B_R satisfies the estimates

$$\frac{2\pi c}{\ln(R/\ell)} \le \operatorname{cap}_2(\Sigma_\ell; B_R) \le \frac{2\pi}{\ln(R/\ell)},$$

where the constant $c > 0$ in the lower bound is dimensional.

Proof. Consider the following arc of the circumference $\partial B_{2\ell}$

$$A_\ell := \Big\{(x, y) \ : \ -\ell < x < \ell, \ y = -\sqrt{4\ell^2 - x^2}\Big\},$$

and take functions $\eta_\ell : \mathbb{R} \to \mathbb{R}$ and $\phi_\ell : \mathbb{R} \to \mathbb{R}$ such that:

$$\begin{gathered}
\eta_\ell(x) = \sqrt{4\ell^2 - x^2} \quad \text{for all} \quad x \in [-\ell, \ell],\\
\eta_\ell \in C^1_c((-4\ell, 4\ell)) \quad \text{and} \quad |\nabla \eta_\ell(x)| \le 1 \quad \text{for} \quad x \in (-4\ell, 4\ell);\\
\phi_\ell(x) = 1 \quad \text{for all} \quad x \in [-3\ell, 3\ell],\\
\phi_\ell \in C^1_c((-1, 1)) \quad \text{and} \quad |\nabla \phi_\ell(y)| \le 2 \quad \text{for} \quad y \in (-1, 1).
\end{gathered}$$

Consider the map

$$\Phi_\ell(x, y) = (x, y - \phi_\ell(y)\eta_\ell(x)).$$

Then, Φ_ℓ is a C^1 diffeomorphism from B_R into itself such that $\Phi_\ell(\Sigma_\ell) = A_\ell$. Since

$$J\Phi_\ell(x, y) = \begin{pmatrix} 1 & -\phi_\ell(y)\eta_\ell'(x) \\ 0 & 1 - \phi_\ell'(y)\eta_\ell(x) \end{pmatrix} \qquad \text{and} \qquad J\Phi_\ell(x, y)^{-1} = \begin{pmatrix} 1 & \frac{\phi_\ell(y)\eta_\ell'(x)}{1-\phi_\ell'(y)\eta_\ell(x)} \\ 0 & \frac{1}{1-\phi_\ell'(y)\eta_\ell(x)} \end{pmatrix},$$

using the smooth deformations estimate from Proposition 8, we get that

$$\operatorname{cap}_2(\Sigma_\ell; B_R) \le C \operatorname{cap}_2(A_\ell; B_R),$$

where C is a dimensional constant. Now, we notice that the arc A_ℓ is exactly 1/6 of the circle $\partial B_{2\ell}$. Thus, by rotating A_ℓ, we get 6 arcs $A_\ell^{(j)}$, $j = 1, \dots, 6$, such that

$$\operatorname{cap}_2(A_\ell; B_R) = \operatorname{cap}_2(A_\ell^{(j)}; B_R) \quad \text{for all} \quad j = 1, \dots, 6,$$

and

$$\partial B_{2\ell} = \bigcup_{j=1}^{6} A_\ell^{(j)}.$$

Now, the subadditivity of the capacity gives

$$\operatorname{cap}_2(\partial B_{2\ell}; B_R) \le \sum_{j=1}^{6} \operatorname{cap}_2(A_\ell^{(j)}; B_R) = 6 \operatorname{cap}_2(A_\ell; B_R) \le 6C \operatorname{cap}_2(\Sigma_\ell; B_R).$$

Finally, thanks to Proposition 10, we get the claim. $\square$

THE RELATIVE p-CAPACITY ALONG MONOTONE SEQUENCES

In this section we will prove the following proposition

Proposition 12 (Relative capacity along increasing sequences of sets)**.** *Let $d \ge 2$ and $p \in (1, +\infty)$. Let Ω be a bounded open set in $\mathbb{R}^d$ and $A \Subset \Omega$. Let $(A_n)_{n\ge1}$ be an increasing sequence of sets in $\mathbb{R}^d$ such that*

$$A = \bigcup_{n=1}^{+\infty} A_n.$$

Then,

$$\operatorname{cap}_p(A; \Omega) = \lim_{n\to+\infty} \operatorname{cap}_p(A_n; \Omega) = \sup_{n\to+\infty} \operatorname{cap}_p(A_n; \Omega).$$

Proof. Since A_n is an increasing sequence, we have the equality

$$\lim_{n\to+\infty} \operatorname{cap}_p(A_n;\Omega) = \sup_{n\geq 1} \left\{\operatorname{cap}_p(A_n;\Omega)\right\}.$$

Moreover, since A contains A_n, we have

$$\operatorname{cap}_p(A;\Omega) \geq \operatorname{cap}_p(A_n;\Omega) \quad \text{for all} \quad n\geq 1.$$

Combining these two observations, we get that

$$\operatorname{cap}_p(A;\Omega) \geq \lim_{n\to+\infty} \operatorname{cap}_p(A_n;\Omega).$$

In order to show that an equality holds, we only need to prove the opposite inequality

$$\operatorname{cap}_p(A;\Omega) \leq \lim_{n\to+\infty} \operatorname{cap}_p(A_n;\Omega).$$

We will show that, for any fixed $\varepsilon > 0$, it holds

$$\operatorname{cap}_p(A;\Omega) \leq \lim_{n\to+\infty} \operatorname{cap}_p(A_n;\Omega) + \varepsilon.$$

Since,

$$\operatorname{cap}_p({}_nA;\Omega) < +\infty \quad \text{for all} \quad n\geq 1,$$

we can find a function $u_n \in W^{1,p}_0(\Omega)$ satisfying the conditions

- $0 \leq u_n \leq 1$ on Ω,
- $u_n = 1$ on an open set Ω_n containing A_n,

and such that

$$\operatorname{cap}_p(A_n : \Omega) \leq \int_\Omega |\nabla u_n|^p\,dx \leq \operatorname{cap}_p(A_n;\Omega) + \frac{\varepsilon}{2^n}. \tag{2}$$

We next define the sequence of functions

$$h_n := u_1 \vee u_2 \vee \cdots \vee u_n.$$

Since the functions h_n can be obtained inductively via the relations

$$h_1 = u_1 \quad \text{and} \quad h_n = h_{n-1} \vee u_n \quad \text{for all} \quad n \geq 2,$$

we get that, for all $n \geq 2$,

- $h_n \in W^{1,p}_0(\Omega)$;
- $0 \leq h_n \leq 1$ on Ω;
- $h_{n-1} \leq h_n$ on Ω;
- $h_n = 1$ on the open set $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n$, which contains A_n.

We recall that $u_n \vee h_{n-1} \in W^{1,p}_0(\Omega)$, $u_n \wedge h_{n-1} \in W^{1,p}_0(\Omega)$ and that we have the formula

$$\int_\Omega |\nabla(u_n \vee h_{n-1})|^p\,dx + \int_\Omega |\nabla(u_n \wedge h_{n-1})|^p\,dx = \int_\Omega |\nabla u_n|^p\,dx + \int_\Omega |\nabla h_{n-1}|^p\,dx. \tag{3}$$

Since

$$u_n \wedge h_{n-1} \geq 1 \quad \text{on the open set} \quad \Omega_n \cap \left(\bigcup_{k=1}^{n-1} \Omega_k\right),$$

which still contains A_{n-1}, by the definition of the relative capacity, we have

$$\operatorname{cap}_p(A_{n-1};\Omega) \leq \int_\Omega |\nabla(u_n \wedge h_{n-1})|^p\,dx.$$

Plugging this into the formula (3), we obtain the inequality

$$\int_\Omega |\nabla(u_n \vee h_{n-1})|^p\,dx + \operatorname{cap}_p(A_{n-1};\Omega) \leq \int_\Omega |\nabla u_n|^p\,dx + \int_\Omega |\nabla h_{n-1}|^p\,dx,$$

which combined with the fact that u_n was chosen in such a way that the inequality (2) is satisfied, gives

$$\int_\Omega |\nabla h_n|^p\,dx + \operatorname{cap}_p(A_{n-1};\Omega) \leq \operatorname{cap}_p(A_n;\Omega) + \frac{\varepsilon}{2^n} + \int_\Omega |\nabla h_{n-1}|^p\,dx,$$

for every $n \geq 2$. Now, summing up these inequalities for $n \geq 2$, together with

$$\int_\Omega |\nabla h_1|^p\,dx \leq \operatorname{cap}_p(A_1;\Omega) + \frac{\varepsilon}{2},$$

we get that

$$\int_\Omega |\nabla h_n|^p\,dx \le \mathrm{cap}_p(A_n;\Omega) + \varepsilon\sum_{k=1}^{n}\frac{1}{2^k} \le \mathrm{cap}_p(A_n;\Omega) + \varepsilon\,,$$

which implies the uniform bound

$$\int_\Omega |\nabla h_n|^p\,dx \le \mathrm{cap}_p(A;\Omega) + \varepsilon \quad \text{for all} \quad n\ge 1. \tag{4}$$

Now, by construction, the sequence h_n is monotone increasing and bounded:

$$0\le h_n\le 1 \quad \text{on} \quad \Omega.$$

By the monotone convergence theorem, h_n converges strongly in $L^p(\Omega)$ to the function

$$h(x) := \sup_{n\ge1} h_n(x).$$

Moreover, thanks to the uniform bound (4), we have that $h\in W^{1,p}_0(\Omega)$ and that the sequence of weak gradients ∇h_n converges weakly in L^p to ∇h (here we use that $p>1$). By the semicontinuity of the norm of the gradient with respect to the weak convergence, we get

$$\int_\Omega |\nabla h|^p\,dx \le \liminf_{n\to+\infty}\int_\Omega |\nabla h_n|^p\,dx \le \varepsilon + \lim_{n\to+\infty}\mathrm{cap}_p(A_n;\Omega).$$

We next notice that, by construction, we have:

- $0\le h\le 1$ on Ω;
- $h=1$ on the union of the open set $\bigcup_{n\ge1}\Omega_n$, which contains A.

Thus, by the definition of the relative p-capacity, we have

$$\mathrm{cap}_p(A;\Omega)\le \int_\Omega |\nabla h|^p\,dx,$$

so we get that

$$\mathrm{cap}_p(A;\Omega)\le \varepsilon + \lim_{n\to+\infty}\left\{\mathrm{cap}_p(A_n;\Omega)\right\},$$

which concludes the proof. □

OUTER MEASURE PROPERTY OF THE RELATIVE p-CAPACITY

From Proposition 12, we immediately obtain

Proposition 13 (σ-subadditivity of the relative p-capacity)**.** *Let $p\in(1,+\infty)$ and $d\ge2$. Let Ω be a bounded open set in $\mathbb{R}^d$. Suppose that $(A_n)_{n\ge1}$ is a sequence of sets in $\mathbb{R}^d$ such that $A_n\Subset\Omega$ and suppose that $A=\cup_n A_n$ also satisfies the inclusion $A\Subset\Omega$. Then,*

$$\mathrm{cap}_p(A;\Omega)\le\sum_{n=1}^{+\infty}\mathrm{cap}_p(A_n;\Omega).$$

Proof. Consider the sequence of sets

$$\widetilde{A}_n := \bigcup_{k=1}^{n} A_k.$$

Thanks to the subadditivity of the capacity we have

$$\mathrm{cap}_p(\widetilde{A}_n;\Omega)\le\sum_{k=1}^{n}\mathrm{cap}_p(A_k;\Omega).$$

Now, since $\widetilde{A}_n$ is an increasing sequence and since

$$A=\bigcup_{n=1}^{\infty}\widetilde{A}_n,$$

thanks to Proposition 12, we can take the limit on the left-hand side obtaining

$$\mathrm{cap}_p(A;\Omega)=\lim_{n\to+\infty}\mathrm{cap}_p(\widetilde{A}_n;\Omega)\le\sum_{n=1}^{+\infty}\mathrm{cap}_p(A_n;\Omega).$$ □