Sets of zero capacity

We are mainly interested in the case $p \in (1, d]$ in which the relative capacity allows to define a non-trivial family of sets of zero capacity in $\mathbb{R}^d$. Precisely, we will say that a set $A \subset \mathbb{R}^d$ has null capacity if

$$\mathrm{cap}_p\big(A \cap B_r(x); B_{2r}(x)\big) = 0 \quad \text{for all balls} \quad B_r(x) \subset \mathbb{R}^d. \tag{1}$$

The relative capacity turns out to be a more suitable tool for defining sets of null capacity. Indeed, we will show that in the subcritical case $p < d$, we have

$$\text{“}A \text{ has null capacity”} \quad \Leftrightarrow \quad \mathrm{cap}_p(A) = 0,$$

so this definition is consistent with the classical definition of sets of null capacity in $\mathbb{R}^d$, which is discussed in detail in the book by Evans and Gariepy. In the critical case $p = d \geq 2$, on the other hand, all bounded sets $A \subset \mathbb{R}^d$ are such that $\mathrm{cap}_d(A) = 0$, while definition (1) produces a non-trivial class of sets of null capacity. Finally, we notice that the theory we develop for the sets of zero capacity (defined through (1)) allows to treat all cases $p \in (1, d]$ at once.

Definition

Definition 1 (Sets of zero p-capacity). *Let $d \geq 2$, $p \in [1, +\infty)$.*
We say that a set $A \subset \mathbb{R}^d$ has zero p-capacity if

$$\mathrm{cap}_p\big(A \cap B_r(x); B_{2r}(x)\big) = 0 \quad \textit{for all balls} \quad B_r(x) \subset \mathbb{R}^d.$$

Elementary properties of the sets of zero capacity

Proposition 2 (Finite unions of sets of zero capacity). *Let $p \in [1, +\infty)$ and $d \geq 2$. If A is a set of zero p-capacity in $\mathbb{R}^d$, then every $\Omega \subset A$ is also a set of zero p-capacity in $\mathbb{R}^d$.*

Proof. This follows immediately from the monotonicity of the relative capacity.

$$\mathrm{cap}_p\big(\Omega \cap B_r(x); B_{2r}(x)\big) \leq \mathrm{cap}_p\big(A \cap B_r(x); B_{2r}(x)\big),$$

which holds for all balls $B_r(x)$ in $\mathbb{R}^d$. □

Proposition 3 (Finite unions of sets of zero capacity). *Let $p \in [1, +\infty)$ and $d \geq 2$. If A_1 and A_2 are sets of zero p-capacity in $\mathbb{R}^d$, then $A_1 \cup A_2$ is also a set of zero p-capacity in $\mathbb{R}^d$.*

Proof. This follows from the inequality

$$\mathrm{cap}_p\big((A_1 \cup A_2) \cap B_r(x); B_{2r}(x)\big) \leq \mathrm{cap}_p\big(A_1 \cap B_r(x); B_{2r}(x)\big) + \mathrm{cap}_p\big(A_2 \cap B_r(x); B_{2r}(x)\big),$$

which holds for all balls $B_r(x)$ in $\mathbb{R}^d$. □

Proposition 4 (Infinite unions of sets of zero capacity). *Let $p \in (1, +\infty)$ and $d \geq 2$. If $(A_n)_{n\geq 1}$ is a sequence of sets of zero p-capacity in $\mathbb{R}^d$, then the union $A := \bigcup_{n\geq 1} A_n$ is also a set of zero p-capacity in $\mathbb{R}^d$.*

Proof. This follows from the inequality

$$\mathrm{cap}_p\big(A \cap B_r(x); B_{2r}(x)\big) \leq \sum_{n=1}^{+\infty} \mathrm{cap}_p\big(A_n \cap B_r(x); B_{2r}(x)\big),$$

which holds for all balls $B_r(x)$ in $\mathbb{R}^d$. □

Sets of zero capacity in the case $p > d$

In this section, we show that, when $p > d$, the only set of zero capacity is the empty set.

Proposition 5. *Let $p > d \geq 2$ and Ω be a bounded open set in $\mathbb{R}^d$.*
Then, for every set non-empty set $A \Subset \Omega$, we have

$$0 < \mathrm{cap}_p(A;\Omega) < +\infty.$$

In particular, when $p > d$, the only set of zero p-capacity in $\mathbb{R}^d$ is the empty set.

Proof. Since $\overline{A} \subset \Omega$ we can find a function $\varphi \in C_c^\infty(\Omega)$ such that $\overline{\varphi} \geq 1$ on $\overline{A}$. This proves that $\mathrm{cap}_p(A;\Omega) < +\infty$. In order to show that $\mathrm{cap}_p(A;\Omega) > 0$, we consider a sequence $u_n \in W_0^{1,p}(\Omega)$ such that $0 \leq u_n \leq 1$ on Ω, $u_n = 1$ in a neighborhood of A, and

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_n|^p\,dx = \mathrm{cap}_p(A;\Omega).$$

Now, thanks to the Sobolev-Morrey embedding $W^{1,p}(\mathbb{R}^d) \subset C^{0,\alpha}(\mathbb{R}^d)$ and the Ascoli-Arzelà's theorem, we get that there is a function $u \in W_0^{1,p}(\Omega)$ such that up to extracting a subsequence, u_n converges to u uniformly on $\overline{\Omega}$ and weakly in $W_0^{1,p}(\Omega)$. The weak convergence gives the bound from below,

$$\int_\Omega |\nabla u|^p\,dx \leq \lim_{n\to+\infty} \int_\Omega |\nabla u_n|^p\,dx = \mathrm{cap}_p(A;\Omega),$$

The uniform convergence on the other hand gives that $u = 0$ on $\partial\Omega$ and $u = 1$ on $\overline{A}$, which guarantees that u is not a constant, so we get the lower bound

$$0 < \int_\Omega |\nabla u|^p\,dx \leq \mathrm{cap}_p(A;\Omega).$$

□

In what follows we will focus on the case $p \leq d$.

The Lebesgue measures of a set of zero capacity

In the next propositions we show that:

- all the sets of zero p-capacity in $\mathbb{R}^d$ have zero Lebesgue measure;
- all the sets of zero p-capacity in $\mathbb{R}^d$, which are contained in a C^1 graph, have zero surface measure.

Proposition 6 (Capacity and Lebesgue measure)**.** *Let $d \geq 2$ and $p \in [1,d]$.*
Suppose that Ω is a bounded open set of $\mathbb{R}^d$ and that $A \Subset \Omega$ is a measurable set with Lebesgue measure

$$\mathcal{L}^d(A) \in (0,+\infty].$$

Then,

$$\mathrm{cap}_p(A;\Omega) > 0.$$

Proof. Let $u \in W_0^{1,p}(\Omega)$ be such that $u = 1$ in a neighborhood of A. There is a constant $C = C(d,p,|\Omega|)$, depending only on d, p and the Lebesgue measure of Ω such that

$$\int_\Omega |u|^p \leq C \int_\Omega |\nabla u|^p\,dx.$$

Thus, since $u \geq 1$ on A we get

$$|A| \leq C \int_\Omega |\nabla u|^p\,dx.$$

Taking the infimum over u we obtain

$$|A| \leq C\,\mathrm{cap}(A;\Omega),$$

which concludes the proof. □

Set of zero capacity contained in a smooth surface

Lemma 7 (Approximation with smooth functions)**.** *Let $d \geq 2$ and $p \in [1, d]$.*
Suppose that Ω is a bounded open set of $\mathbb{R}^d$ and that the set $A \Subset \Omega$ has zero capacity. Then, there is a sequence of functions $w_n \in C_c^\infty(\Omega)$ such that:

- *$0 \leq w_n \leq 1$ in Ω;*
- *A is contained in the interior of the set $\{w_n = 1\}$;*
- *$\lim_{n\to+\infty} \int_\Omega \big(|\nabla w_n|^p + w_n^p\big)\,dx = 0$.*

Proof. We first fix a function $\phi_A \in C_c^\infty(\Omega)$ such that $0 \leq \phi_A \leq 1$ in Ω, and $\phi_A \equiv 1$ in a neighborhood of the closed set $\overline{A}$.

Since $\mathrm{cap}_p(A;\Omega) = 0$, there is a sequence of functions $u_n \in W_0^{1,p}(\Omega)$ such that $0 \leq u_n \leq 1$ on Ω, $u_n = 1$ on a neighborhood of A, and

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_n|^p\,dx = 0.$$

Thanks to the Poincaré inequality in Ω

$$\int_\Omega |u_n|^p\,dx \leq \int_\Omega |\nabla u_n|^p\,dx,$$

we have that

$$\lim_{n\to+\infty} \int_\Omega |u_n|^p\,dx + \int_\Omega |\nabla u_n|^p\,dx = 0.$$

Thus, the sequence of functions

$$v_n := \phi_A u_n,$$

is such that: $v_n \in W_0^{1,p}(\Omega)$, $0 \leq u_n \leq 1$ on Ω, $u_n = 1$ on a neighborhood of A, and

$$\lim_{n\to+\infty} \int_\Omega (|\nabla v_n|^p + v_n^p)\,dx = 0.$$

Moreover, since $v_n = 0$ where $\phi = 0$, we have that the set $\{v_n \neq 0\}$ is compactly supported in Ω.
Now, choose a function

$$\phi \in C_c^\infty(\mathbb{R}^d), \quad \phi \equiv 1 \ \text{ on } \ B_1, \quad \int_{\mathbb{R}^d} \phi(x)\,dx = 1,$$

and set

$$\phi_r(x) := \frac{1}{r^d}\phi(x/r).$$

For a fixed $n > 1$, consider the family of convolutions

$$\phi_r * v_n, \ \ r > 0.$$

Since the support of v_n is compactly contained in Ω, we have that $\phi_r * v_n$ converges strongly in $W_0^{1,p}(\Omega)$ to v_n as $r \to 0$. Thus, for each n, we can choose $r_n > 0$ small enough such that:

- $w_n := \phi_{r_n} * v_n \in C_c^\infty(\Omega)$
- $0 \leq w_n \leq 1$ in Ω;
- A is contained in the interior of the set $\{w_n = 1\}$;
- $\int_\Omega \big(|\nabla w_n|^p + w_n^p\big)\,dx \leq \int_\Omega \big(|\nabla v_n|^p + v_n^p\big)\,dx + \frac{1}{n}$,

which concludes the proof. □

Proposition 8 (Capacity and surface measure)**.** *Let $d \geq 2$ and $p \in [1, d]$.*
Suppose that Ω is a bounded open set of $\mathbb{R}^d$ and that the set $A \Subset \Omega$ is of the form

$$\Gamma(\eta; A') = \big\{(x', \eta(x')) \ : \ x' \in A'\big\},$$

where $A' \subset \mathbb{R}^{d-1}$ an where $\eta : \mathbb{R}^{d-1} \to \mathbb{R}$ is a C^1 function. Then, if $\mathrm{cap}_p(A;\Omega) = 0$, we have that $\mathcal{L}^{d-1}(A') = 0$. In particular, this implies that if $\mathrm{cap}_p(A;\Omega) = 0$, then the surface measure σ of A, defined as

$$\sigma(A) := \int_{A'} \sqrt{1 + |\nabla_{x'}\eta(x')|^2}\,dx',$$

also vanishes.

Proof. Let $x_0' \in A' \subset \mathbb{R}^{d-1}$ be such that

$$|B_r'(x_0') \cap A'| > 0 \quad \text{for all} \quad r > 0.$$

Choose $r > 0$ small enough in such a way that the closure of the set

$$C_r^+ := \Big\{(x', x_d) \ : \ x' \in B_r'(x_0),\ \eta(x') < x_d < \eta(x') + r\Big\}$$

is contained in Ω.

Since $\text{cap}_p(A;\Omega) = 0$, Lemma 7 implies that there is a sequence of smooth functions $w_n \in C_c^\infty(\Omega)$ such that $0 \le w_n \le 1$ on Ω, $w_n = 1$ on a neighborhood of A, and

$$\lim_{n\to+\infty} \int_\Omega \Big(|\nabla w_n|^p + w_n^p\Big)\,dx = 0.$$

Now, the trace inequality in C_r^+ implies that we can estimate the norm of w_n on A from above as follows. For each $x' \in B_r'(x_0)$ and each $y \in (0, r)$ we have

$$\begin{aligned} w_n^2(x', \eta(x')) &\le w_n^2(x', y) + \int_{\eta(x')}^{\eta(x')+y} 2w_n\big(x', \eta(x') + t\big)\partial_d w_n\big(x', \eta(x') + t\big)\,dt \\ &\le w_n^2(x', y) + \int_{\eta(x')}^{\eta(x')+r} 2|w_n|\big(x', \eta(x') + t\big)|\nabla w_n|\big(x', \eta(x') + t\big)\,dt. \end{aligned}$$

Integrating for $y \in \big(\eta(x'), \eta(x') + r\big)$ we get

$$r w_n^2(x', \eta(x')) \le \int_{\eta(x')}^{\eta(x')+r} w_n^2(x', t)\,dt + \int_{\eta(x')}^{\eta(x')+y} 2w_n\big(x', \eta(x') + t\big)|\nabla w_n|\big(x', \eta(x') + t\big)\,dt,$$

an finally, integrating in x' an using the inequality $2ab \le a^2 + b^2$, we get

$$r \int_{B_r'(x_0')} w_n^2(x', \eta(x'))\,dx' \le 2 \int_{C_r^+} w_n^2(x)\,dx + \int_{C_r^+} |\nabla w_n|^2(x)\,dx.$$

Now, since $w_n \ge 1$ on A', we get

$$r\mathcal{L}^{d-1}(A') \le 2 \int_{C_r^+} w_n^2(x)\,dx + \int_{C_r^+} |\nabla w_n|^2(x)\,dx.$$

Taking the limit as $n \to +\infty$, we get $\mathcal{L}^{d-1}(A') = 0$. □

A characterization of the sets of zero capacity

The next proposition gives a working criteria for determining whether a set $A \subset \mathbb{R}^d$ has zero capacity.

Proposition 9. *Let $d \ge 2$ and $p \in [1, d]$. If the set A can be covered with at most countably many balls*

$$B_{R_k}(x_k), \quad k \ge 1,$$

such that

$$\text{cap}\,\big(A \cap B_{R_k}(x_k); B_{2R_k}(x_k)\big) = 0 \quad \textit{for all} \quad k \ge 1,$$

then A has zero capacity in the sense of Definition 1.

Proof. Fix a ball $B_R(x_0) \subset \mathbb{R}^d$. It is sufficient to show that

$$\text{cap}\,\big(A \cap B_R(x_0); B_{2R}(x_0)\big) = 0.$$

Without loss of generality we set $x_0 = 0$. Since

$$A = \bigcup_{k\ge1} A \cap B_{R_k}(x_k),$$

it is sufficient to show that

$$\text{cap}\,\big(A \cap B_{R_k}(x_k) \cap B_R; B_{2R}\big) = 0.$$

Now, fix $\varepsilon > 0$ and take a function $u_k \in W_0^{1,p}(B_{2R_k}(x_k))$ such that

$$\int \Big(|\nabla u_k|^p + u_k^p\Big)\,dx \le \varepsilon \quad \text{and} \quad u_k = 1 \quad \text{in a neighborhood of} \quad A \cap B_{R_k}(x_k).$$

Moreover, let $\varphi_R \in W_0^{1,p}(B_{2R})$ be a function such that

$$\varphi_R = 1 \quad \text{on} \quad B_R \ , \qquad 0 \le \varphi_R \le 1 \quad \text{and} \quad |\nabla \varphi_R| \le \frac{2}{R} \quad \text{on} \quad B_{2R} \setminus B_R \ .$$

Then,

$$\begin{aligned} \operatorname{cap}\big(A \cap B_{R_k}(x_k) \cap B_R; B_{2R}\big) &\le \int_{B_{2R}} |\nabla(u_k \varphi_R)|^p \, dx \\ &\le C_p \int_{B_{2R}} |\nabla u_k|^p + \frac{2^p C_p}{R^p} \int_{B_{2R}} u_k^p \, dx \\ &\le C_p \left(1 + \frac{2^p}{R^2}\right) \varepsilon, \end{aligned}$$

where C_p is a constant depending only on p. Since $\varepsilon > 0$ is arbitrary, we get that

$$\operatorname{cap}\big(A \cap B_{R_k}(x_k) \cap B_R; B_{2R}\big) = 0,$$

which concludes the proof. □

Corollary 10. *Let $d \ge 2$ and $p \in [1, d]$. Then, all the points $x_0 \in \mathbb{R}^d$ have zero capacity.*

Corollary 11. *Let $d \ge 2$ and $p \in [1, d]$. The set $\mathbb{Q}^d \subset \mathbb{R}^d$ of all points with rational coordinates has zero capacity.*

On the definition of a set of zero capacity

The next proposition shows that the definition of a set of zero capacity (Definition 1), which is given in terms of the relative capacity, is consistent with the natural definition of a set of zero capacity with respect to the capacity in $\mathbb{R}^d$ (defined in the previous sections).

Proposition 12. *Let $d \ge 2$ and $p \in [1, d)$, and let $A \subset \mathbb{R}^d$. Then, the following holds:*

(i) *If $\operatorname{cap}_p(A) = 0$, then the set A has zero capacity in the sense of Definition 1.*
(ii) *If A has zero capacity in the sense of Definition 1, then $\operatorname{cap}_p(A) = 0$.*

Proof. In order to prove (i), it is sufficient to show that for any ball $B_R \subset \mathbb{R}^d$

$$\operatorname{cap}_p(A \cap B_R, B_{2R}) = 0.$$

Fix $\varphi_R \in C_c^\infty(B_{2R})$ such that

$$\varphi_R = 1 \quad \text{on} \quad B_R \ , \qquad 0 \le \varphi_R \le 1 \quad \text{and} \quad |\nabla \varphi_R| \le \frac{2}{R} \quad \text{on} \quad B_{2R} \setminus B_R \ .$$

Since $\operatorname{cap}_p(A) = 0$, we can find a sequence of functions $u_n \in W^{1,p}(\mathbb{R}^d)$ such that

$$u_n = 1 \quad \text{in a neighborhood of} \quad A \ , \qquad 0 \le u_n \le 1 \quad \text{and} \quad \lim_{n \to +\infty} \int_{\mathbb{R}^d} |\nabla u_n|^p \, dx = 0.$$

Consider the sequence of functions $v_n := \varphi_R u_n$. Then $v_n \in W_0^{1,p}(B_{2R})$ and

$$v_n = 1 \quad \text{in a neighborhood of} \quad A \cap B_R \quad \text{and} \quad 0 \le v_n \le 1 \quad \text{on} \quad B_{2R}.$$

Moreover, by using the Gagliardo-Nirenberg-Sobolev's inequality

$$\left(\int_{\mathbb{R}^d} u_n^{\frac{pd}{d-p}} \, dx\right)^{\frac{d-p}{d}} \le C_{p,d} \int_{\mathbb{R}^d} |\nabla u_n|^p \, dx,$$

we can compute

$$\begin{aligned}\int_{B_{2R}} |\nabla v_n|^p \, dx &\le C_p \int_{B_{2R}} |\nabla v_n|^p + \frac{2^p C_p}{R^p} \int_{B_{2R}} u_n^p \, dx \\ &\le C_p \int_{B_{2R}} |\nabla v_n|^p + \frac{2^p C_p}{R^p} |B_{2R}|^{\frac{p}{d}} \left(\int_{B_{2R}} u_n^{\frac{dp}{d-p}} \, dx \right)^{\frac{d-p}{p}} \\ &\le C_p \int_{B_{2R}} |\nabla v_n|^p + \frac{2^p C_p}{R^p} |B_{2R}|^{\frac{p}{d}} C_{p,d} \int_{\mathbb{R}^d} |\nabla u_n|^p \, dx \\ &= C_p \left(1 + \frac{2^p C_p}{R^p} |B_{2R}|^{\frac{p}{d}} C_{p,d} \right) \int_{\mathbb{R}^d} |\nabla u_n|^p \, dx \\ &= C_{p,d} \int_{\mathbb{R}^d} |\nabla u_n|^p \, dx,\end{aligned}$$

where by $C_{p,d}$ is a constant that depends only on d, p (as usual, $C_{p,d}$ might changes from line to line). By taking $n \to +\infty$, we get that

$$\mathrm{cap}_p(A \cap B_R, B_{2R}) = 0,$$

which concludes the proof of (i).

We next prove (ii). Suppose that $A \subset \mathbb{R}^d$ has zero capacity in the sense of Definition 1. Then, for any $R > 0$ we have

$$\mathrm{cap}_p(A \cap B_R; B_{2R}) = 0.$$

Now, notice that by the definitions of the relative capacity $\mathrm{cap}_p(A \cap B_R; B_{2R})$ and the capacity $\mathrm{cap}_p(A \cap B_R)$ in the whole space $\mathbb{R}^d$, we have

$$\mathrm{cap}_p(A \cap B_R) \le \mathrm{cap}_p(A \cap B_R; B_{2R}).$$

Thus,

$$\mathrm{cap}_p(A \cap B_R) = 0 \quad \text{for all} \quad R > 0.$$

Since the family of sets $A \cap B_R$ is increasing, taking the limit as $R \to +\infty$, we get

$$\mathrm{cap}_p(A) = \lim_{R \to +\infty} \mathrm{cap}_p(A \cap B_R) = 0,$$

which concludes the proof. $\square$