Abstract theorems from measure theory

Carathéodory's criterion

Theorem 1. *Let X be a given set and let $\mathcal{P}(X)$ be the family of all subsets of X. Let $\mu : \mathcal{P}(X) \to [0, +\infty]$ be a function with the following properties:*

(a) $\mu(\emptyset) = 0$;
(b) *if* $E_1 \subset E_2$, *then* $\mu(E_1) \le \mu(E_2)$;
(c) *if* $\{E_i\}_{i\ge1}$ *is a countable family of subsets of* X, *then*

$$\mu(E) \le \sum_{i=1}^{+\infty} \mu(E_i) \qquad \textit{where} \qquad E = \bigcup_{i=1}^{+\infty} E_i\,.$$

Let $\mathcal{A}$ be the family of all subsets E of X with the following property:

$$\mu(F) = \mu(F\cap E) + \mu(F\setminus E) \quad \textit{for all} \quad F \subset X.$$

Then, we have the following:

(i) $\mathcal{A}$ *is a σ-algebra, that is:*
  (i.1) $\emptyset \in \mathcal{A}$ *and* $X \in \mathcal{A}$;
  (i.2) *if* $E \in \mathcal{A}$, *then also* $X \setminus E \in \mathcal{A}$;
  (i.3) *if* $E_1, E_2 \in \mathcal{A}$, *then also* $E_2 \cap E_2,\ E_1 \cup E_2,\ E_1 \setminus E_2,\ E_2 \setminus E_1 \in \mathcal{A}$;
  (i.4) *if* $\{E_i\}_{i\ge1}$ *is a family of sets such that* $E_i \in \mathcal{A}$, *then*

$$\bigcap_{i=1}^{+\infty} E_i \in \mathcal{A} \qquad \textit{and} \qquad \bigcup_{i=1}^{+\infty} E_i \in \mathcal{A}\,.$$

(ii) $\mathcal{A}$ *contains all sets of zero measure: if* $\mu(E) = 0$, *then* $E \in \mathcal{A}$;

(iii) μ *is a σ-additive measure on* $\mathcal{A}$, *that is:*
  (ii.1) *if* $E_1, E_2 \in \mathcal{A}$ *are disjoint sets, then*

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2);$$

  (ii.2) $\{E_i\}_{i\ge1}$ *is a countable family of disjoint sets* $E_i \in \mathcal{A}$, *then*

$$\mu(E) = \sum_{i=1}^{+\infty} \mu(E_i) \qquad \textit{where} \qquad E = \bigcup_{i=1}^{+\infty} E_i\,.$$

Proof. We proceed in several steps.

Step 1. Proof of (i.1), (i.2) and (i.3). First of all, we notice that since

$$\mu(F\cap\emptyset) = \mu(\emptyset) = 0 \quad \text{and} \quad \mu(F\setminus\emptyset) = \mu(F),$$

we have that

$$\mu(F) = \mu(F\cap\emptyset) + \mu(F\setminus\emptyset) \quad \text{for all} \quad F \subset X,$$

so $\emptyset \in \mathcal{A}$. Furthermore, since

$$F\cap E = F\setminus(X\setminus E) \quad \text{and} \quad F\setminus E = F\cap(X\setminus E),$$

we have that for all $F, E \subset X$ it holds

$$\mu(F\cap E) + \mu(F\setminus E) = \mu(F\setminus(X\setminus E)) + \mu(F\cap(X\setminus E)),$$

which implies

$$E \in \mathcal{A} \quad \Leftrightarrow \quad X\setminus E \in \mathcal{A}.$$

This concludes the proof of the first two bullets in (i). We next consider two sets $E_1, E_2 \in \mathcal{A}$. We will show that $E_1 \cup E_2 \in \mathcal{A}$. By the definition of $\mathcal{A}$ we have

$$\mu(F) = \mu(F\cap E_1) + \mu(F\setminus E_1) \quad \text{for all} \quad F \subset X, \tag{1}$$

$$\mu(F) = \mu(F\cap E_2) + \mu(F\setminus E_2) \quad \text{for all} \quad F \subset X, \tag{2}$$

We start from

$$\begin{aligned}\mu(F) &= \mu(F\cap E_1)+\mu(F\setminus E_1) && \text{(testing (1) with } F)\\ &= \mu\Big((F\cap E_1)\cap E_2\Big)+\mu\Big((F\cap E_1)\setminus E_2\Big) && \text{(testing (2) with } F\cap E_1)\\ &\quad+\mu\Big((F\setminus E_1)\cap E_2\Big)+\mu\Big((F\setminus E_1)\setminus E_2\Big) && \text{(testing (2) with } F\setminus E_1).\end{aligned}$$

Now since

$$F\cap(E_1\cup E_2)=\Big(F\cap E_1\cap E_2\Big)\cup\Big(F\cap E_1\setminus E_2\Big)\cup\Big(F\cap E_2\setminus E_1\Big),$$

the subadditivity of μ gives

$$\mu\Big(F\cap(E_1\cup E_2)\Big)\le\mu\Big(F\cap E_1\cap E_2\Big)+\mu\Big(F\cap E_1\setminus E_2\Big)+\mu\Big(F\cap E_2\setminus E_1\Big),$$

so we get

$$\begin{aligned}\mu(F)&=\Big[\mu\Big((F\cap E_1)\cap E_2\Big)+\mu\Big((F\cap E_1)\setminus E_2\Big)+\mu\Big((F\setminus E_1)\cap E_2\Big)\Big]+\mu\Big((F\setminus E_1)\setminus E_2\Big)\\ &\ge\mu\Big(F\cap(E_1\cup E_2)\Big)+\mu\Big(F\setminus(E_1\cup E_2)\Big).\end{aligned}$$

Moreover, using again the subadditivy of μ we get the converse inequality,

$$\mu(F)\le\mu\Big(F\cap(E_1\cup E_2)\Big)+\mu\Big(F\setminus(E_1\cup E_2)\Big),$$

which implies that

$$\mu(F)=\mu\Big(F\cap(E_1\cup E_2)\Big)+\mu\Big(F\setminus(E_1\cup E_2)\Big).$$

Since $F\subset X$ is arbitrary we get that $E_1\cup E_2\in\mathcal{A}$. Next, since

$$E_1\cap E_2=X\setminus\Big((X\setminus E_1)\cup(X\setminus E_2)\Big),$$

and

$$E_1\setminus E_2=E_1\cap(X\setminus E_2),$$

we conclude the proof of (i.3).

Step 2. Proof of (ii). Let $\mu(E)=0$. Let $F\subset X$ be an arbitrary subset of X. Since $E\cap F\subset E$ and $F\setminus E\subset F$ the monotonicity of μ implies

$$\mu(F\cap E)+\mu(F\setminus E)\le\mu(E)+\mu(F)=\mu(F).$$

On the other hand, the subadditivity of μ gives

$$\mu(F\cap E)+\mu(F\setminus E)\ge\mu(F).$$

Combining the two inequalities, we obtain that

$$\mu(F\cap E)+\mu(F\setminus E)=\mu(F),$$

and since F is arbitrary, we get that $E\in\mathcal{A}$.

Step 3. Proof of (iii.1) and (iii.2). We first notice that if E_1 and E_2 are disjoint sets in $\mathcal{A}$, then

$$\begin{aligned}\mu(E_2\cup E_1)&=\mu\Big((E_2\cup E_1)\cap E_1\Big)+\mu\Big((E_2\cup E_1)\setminus E_1\Big)\\ &=\mu(E_1)+\mu(E_2).\end{aligned}$$

This proves (iii.1). We next consider a countable family of disjoint sets $\{E_i\}_{i\ge1}$, with $E_i\in\mathcal{A}$, and its union

$$E:=\bigcup_{i\ge1}E_i.$$

For every $n\ge1$ we set

$$\widetilde{E}_n:=\bigcup_{i=1}^n E_i.$$

By (iii.1) we have that

$$\mu(\widetilde{E}_n)=\sum_{i=1}^n\mu(E_i).$$

The monotonicity of μ then implies
$$\mu(E) \geq \mu(\widetilde{E}_n) = \sum_{i=1}^{n} \mu(E_i),$$
so we get
$$\mu(E) \geq \sum_{i=1}^{+\infty} \mu(E_i).$$
On the other hand, the subadditivity of μ implies that
$$\mu(E) \leq \sum_{i=1}^{+\infty} \mu(E_i),$$
so we finally get
$$\mu(E) = \sum_{i=1}^{+\infty} \mu(E_i).$$
Step 4. We claim that if $E_i \in \mathcal{A}$ is a sequence of disjoint sets, then for every subset $F \subset X$ we have
$$\mu(F \cap E) = \sum_{i=1}^{+\infty} \mu(F \cap E_i) \quad \text{where} \quad E := \bigcup_{i=1}^{+\infty} E_i.$$
Indeed, if $E_1 \in \mathcal{A}$ and $E_2 \in \mathcal{A}$ are disjoint sets, then
$$\begin{aligned}\mu\Big((F \cap E_1) \cup (F \cap E_2)\Big) &= \mu\Big(((F \cap E_1) \cup (F \cap E_2)) \cap E_1\Big) + \mu\Big(((F \cap E_1) \cup (F \cap E_2)) \setminus E_1\Big)\\ &= \mu(F \cap E_1) + \mu(F \cap E_2).\end{aligned}$$
Iterating this identity and using the monotonicity of μ, we get
$$\mu(E \cap F) \geq \mu\Big(\bigcup_{i=1}^{n} (F \cap E_i)\Big) = \sum_{i=1}^{n} \mu(F \cap E_i).$$
Passing to the limit as $n \to \infty$, we get the claim.

Step 5. Proof of (i.4). Consider a sequence of sets $E_i \in \mathcal{A}$ and let
$$E := \bigcup_{i=1}^{+\infty} E_i.$$
We define
$$E_1' := E_1 \quad \text{and} \quad E_i' := E_i \setminus (E_1 \cup \cdots \cup E_{i-1}) \quad \text{for every} \quad i \geq 2.$$
By construction:

- the sets E_i' are disjoint;
- $E_i' \in \mathcal{A}$ for all $i \geq 1$;
- for every $n \geq 1$ we have
$$\widetilde{E}_n := \bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} E_i' \qquad \text{and} \qquad \mu(\widetilde{E}_n) = \sum_{i=1}^{n} \mu(E_i').$$

Let now $F \subset X$. Thanks to the subadditivity of μ, in order to show that E is measurable, we only need to prove that
$$\mu(F) \geq \mu(E \cap F) + \mu(F \setminus E).$$
We can assume that $\mu(F) < +\infty$ otherwise the conclusion is trivial. Using the identities
$$E = \bigcup_{i=1}^{+\infty} E_i' \,, \qquad \widetilde{E}_n = \bigcup_{i=1}^{n} E_i' \,, \qquad E \setminus \widetilde{E}_n = \bigcup_{i=n+1}^{+\infty} E_i' \,,$$
we can estimate
$$\begin{aligned}\mu(F \cap E) + \mu(F \setminus E) &\leq \mu(F \cap E) + \mu(F \setminus \widetilde{E}_n) && \text{(by the monotonicity of } \mu)\\ &= \mu(F \cap E) + \mu(F) - \mu(F \cap \widetilde{E}_n) && \text{(since } \widetilde{E}_n \text{ is measurable)}\\ &\leq \mu(F) + \mu(F \cap (E \setminus \widetilde{E}_n)),\end{aligned}$$

where the last inequality is due to the subadditivity of μ:

$$\mu(F \cap E) \leq \mu(F \cap \widetilde{E}_n) + \mu(F \cap (E \setminus \widetilde{E}_n)).$$

Now, using the identity

$$F \cap (E \setminus \widetilde{E}_n) = \bigcup_{i=n+1}^{+\infty} F \cap E_i',$$

and the subadditivity of μ, we get

$$\begin{aligned} \mu(F \cap E) + \mu(F \setminus E) &\leq \mu(F) + \mu(F \cap (E \setminus \widetilde{E}_n)) \\ &\leq \mu(F) + \sum_{i=n+1}^{+\infty} \mu(F \cap E_i'). \end{aligned}$$

Finally, thanks to Step 4 and the assumption $\mu(F \cap E) < +\infty$, we get that

$$\sum_{i=1}^{+\infty} \mu(F \cap E_i') < +\infty,$$

and so

$$\lim_{n \to +\infty} \sum_{i=n+1}^{+\infty} \mu(F \cap E_i') = 0.$$

Thus

$$\mu(F \cap E) + \mu(F \setminus E) \leq \mu(F),$$

which concludes the proof. □

Theorem 2 (Carathéodory's criterion)**.** *Let $d \geq 1$ and let $\mathcal{P}(\mathbb{R}^d)$ be the family of all subsets of $\mathbb{R}^d$. Let $\mu : \mathcal{P}(\mathbb{R}^d) \to [0, +\infty]$ be such that:*

(a) *$\mu(\emptyset) = 0$;*
(b) *if $E_1 \subset E_2$, then $\mu(E_1) \leq \mu(E_2)$;*
(c) *if $\{E_i\}_{i\geq 1}$ is a countable family of subsets of $\mathbb{R}^d$, then*

$$\mu(E) \leq \sum_{i=1}^{+\infty} \mu(E_i) \qquad \textit{where} \qquad E = \bigcup_{i=1}^{+\infty} E_i\,;$$

(d) *μ is additive on the couples of distant sets, precisely*

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2),$$

for any $\emptyset \neq E_1 \subset \mathbb{R}^d$ and $\emptyset \neq E_2 \subset \mathbb{R}^d$ such that

$$\operatorname{dist}(E_1, E_2) := \inf \Big\{ |x_1 - x_2| \ : \ x_1 \in E_1,\ x_2 \in E_2 \Big\} > 0.$$

Let $\mathcal{A}$ be the family of all subsets E of $\mathbb{R}^d$ with the following property:

$$\mu(F) = \mu(F \cap E) + \mu(F \setminus E) \quad \textit{for all} \quad F \subset \mathbb{R}^d.$$

Then, we have the following:

(i) *$\mathcal{A}$ is a σ-algebra, that is:*
 (i.1) *$\emptyset \in \mathcal{A}$ and $\mathbb{R}^d \in \mathcal{A}$;*
 (i.2) *if $E \in \mathcal{A}$, then also $\mathbb{R}^d \setminus E \in \mathcal{A}$;*
 (i.3) *if $E_1, E_2 \in \mathcal{A}$, then also $E_2 \cap E_2,\ E_1 \cup E_2,\ E_1 \setminus E_2,\ E_2 \setminus E_1 \in \mathcal{A}$;*
 (i.4) *if $\{E_i\}_{i\geq 1}$ is a family of sets such that $E_i \in \mathcal{A}$, then*

$$\bigcap_{i=1}^{+\infty} E_i \in \mathcal{A} \qquad \textit{and} \qquad \bigcup_{i=1}^{+\infty} E_i \in \mathcal{A}\ .$$

(ii) *$\mathcal{A}$ contains all sets of zero measure and all open and closed sets, precisely:*
 (ii.1) *if $\mu(E) = 0$, then $E \in \mathcal{A}$;*
 (ii.2) *If E is open, then $E \in \mathcal{A}$;*
 (ii.3) *If E is closed, then $E \in \mathcal{A}$.*

(iii) *μ is a σ-additive measure on $\mathcal{A}$, that is:*

(ii.1) *if $E_1, E_2 \in \mathcal{A}$ are disjoint sets, then*

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2);$$

(ii.2) *$\{E_i\}_{i\geq 1}$ is a countable family of disjoint sets $E_i \in \mathcal{A}$, then*

$$\mu(E) = \sum_{i=1}^{+\infty} \mu(E_i) \qquad \text{where} \qquad E = \bigcup_{i=1}^{+\infty} E_i\,.$$

Since μ satisfies the properties (a), (b), and (c), thanks to Theorem 1, we already know that (i.1)-(i.2)-(i.3)-(i.4), (ii.1), and (iii.1)-(iii.2) do hold. We also notice that thanks to (i.2), we know that (ii.3) implies (ii.2). Thus, we only need to prove (ii.3).

Lemma 3 (σ-additivity over sequences of distant sets)**.** *Let $d \geq 1$ and let $\mu : \mathcal{P}(\mathbb{R}^d) \to [0, +\infty]$ be such that:*

(a) *$\mu(\emptyset) = 0$;*
(b) *if $E_1 \subset E_2$, then $\mu(E_1) \leq \mu(E_2)$;*
(c) *if $\{E_i\}_{i\geq 1}$ is a countable family of subsets of $\mathbb{R}^d$, then*

$$\mu(E) \leq \sum_{i=1}^{+\infty} \mu(E_i) \qquad \text{where} \qquad E = \bigcup_{i=1}^{+\infty} E_i\,;$$

(d) *for all couples of non-empty sets $E_1 \subset \mathbb{R}^d$ and $E_2 \subset \mathbb{R}^d$ satisfying $\operatorname{dist}(E_1, E_2) > 0$, we have*

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2).$$

Suppose that $\{E_i\}_{i\geq 1}$ is a sequence of subsets of $\mathbb{R}^d$ such that

$$\operatorname{dist}(E_i, E_j) > 0 \quad \text{for all} \quad i \neq j.$$

Then,

$$\mu(E) = \sum_{i=1}^{+\infty} \mu(E_i) \qquad \text{where} \qquad E = \bigcup_{i=1}^{+\infty} E_i\,.$$

Proof of Lemma 3. By the property (c) we only need to prove that

$$\mu(E) \geq \sum_{i=1}^{+\infty} \mu(E_i).$$

When $\mu(E) = +\infty$ this trivially holds, so we assume that

$$\mu(E) < +\infty.$$

For every fixed $n \geq 1$ we set

$$\widetilde{E}_n := \bigcup_{k=1}^{n} E_k.$$

Then, by construction, we have

$$\operatorname{dist}(E_n, \widetilde{E}_{n-1}) > 0 \quad \text{for all} \quad n \geq 2,$$

so the property (d) gives

$$\mu(\widetilde{E}_n) = \mu(\widetilde{E}_{n-1}) + \mu(E_n).$$

Thus, we get

$$\mu(\widetilde{E}_n) = \sum_{j=1}^{n} \mu(E_n).$$

Since $\widetilde{E}_n \subset E$, the monotonicity of μ implies that $\mu(E) \geq \mu(\widetilde{E}_n)$, so we get

$$\mu(E) \geq \sum_{j=1}^{n} \mu(E_n),$$

which concludes the proof of Lemma 3. ☐

Proof of Theorem 2 (ii.3). Let $E \subset \mathbb{R}^d$ be a closed set. Consider the distance function

$$d_E : \mathbb{R}^d \to [0, +\infty)\ , \qquad d_E(x) = \inf\Big\{|x-y| \ :\ y \in E\Big\}.$$

We recall the following properties of d_E:

- $d_E(x) = 0$ if and only if $x \in E$;
- for every $x \in \mathbb{R}^d$ there is $x_E \in E$ such that $|x - x_E| = d_E(x)$;
- d_E is 1-Lipschitz, that is, for every $x, y \in \mathbb{R}^d$ we have:

$$|d_E(x) - d_E(y)| \le |x-y| \quad \text{for all} \quad x, y \in \mathbb{R}^d.$$

For every $k \ge 1$ we consider the sets

$$E_k := \Big\{x \in \mathbb{R}^d \ :\ \frac{1}{k+1} < d_E(x) \le \frac{1}{k}\Big\}\ , \qquad \widetilde{E}_k := \Big\{x \in \mathbb{R}^d \ :\ d_E(x) \le \frac{1}{k}\Big\},$$

and we also set

$$E_0 := \Big\{x \in \mathbb{R}^d \ :\ d_E(x) > 1\Big\}.$$

Thanks to the properties of d_E listed above we have that:

- the sets E_k are mutually disjoint;
- the sets E_{2k} are distant from each other; precisely, for every $j \ne k$, we have

$$\inf\Big\{|x-y| \ :\ x \in E_{2k} \ \text{and}\ y \in E_{2j}\Big\} > 0;$$

- the sets E_{2k+1} are distant from each other; precisely, for every $j \ne k$, we have

$$\inf\Big\{|x-y| \ :\ x \in E_{2k+1} \ \text{and}\ y \in E_{2j+1}\Big\} > 0;$$

- the sets E_k are distant from E, precisely:

$$\inf\Big\{|x-y| \ :\ x \in E_k \ \text{and}\ y \in E\Big\} > \frac{1}{k+1};$$

- we have

$$\mathbb{R}^d \setminus E = \bigcup_{j=0}^{+\infty} E_j \qquad \text{and} \qquad \widetilde{E}_k = E \cup \left(\bigcup_{j=k}^{+\infty} E_j\right) \quad \text{for every} \quad k \ge 1.$$

Let F be any subset of $\mathbb{R}^d$. We need to show that

$$\mu(F) = \mu(F \cap E) + \mu(F \setminus E).$$

Thanks to the subadditivity of μ (property (c)), we only need to prove

$$\mu(F) \ge \mu(F \cap E) + \mu(F \setminus E).$$

Since this trivially holds when $\mu(F) = +\infty$, we will proceed under the hypothesis

$$\mu(F) < +\infty.$$

Since

$$\operatorname{dist}(F \cap E, F \setminus \widetilde{E}_n) > 0,$$

we have that

$$\mu(F \cap E) + \mu(F \setminus \widetilde{E}_n) = \mu\Big((F \cap E) \cup (F \setminus \widetilde{E}_n)\Big) \le \mu(F). \tag{3}$$

On the other hand, we have the identity

$$F \setminus E = \Big(F \setminus \widetilde{E}_n\Big) \cup \left(\bigcup_{j=n}^{+\infty} F \cap E_j\right),$$

so thanks to the monotonicity and the subadditivity of μ we get

$$\mu(F \setminus E) \le \mu(F \setminus \widetilde{E}_n) + \sum_{j=n}^{+\infty} \mu(F \cap E_j). \tag{4}$$

Combining (3) and (4) we get that

$$\mu(F \cap E) + \mu(F \setminus E) \leq \mu(F \cap E) + \mu(F \setminus \widetilde{E}_n) + \sum_{j=n}^{+\infty} \mu(F \cap E_j)$$
$$\leq \mu(F) + \sum_{j=n}^{+\infty} \mu(F \cap E_j),$$

so we only need to show that

(5) $$\lim_{n \to +\infty} \sum_{j=n}^{+\infty} \mu(F \cap E_j) = 0.$$

In order to prove that the above limit is zero, it is sufficient to show that

$$\sum_{j=1}^{+\infty} \mu(F \cap E_j) < +\infty$$

But this follows by splitting the sum into even and odd terms and applying Lemma 3. Precisely,

$$\sum_{j=1}^{+\infty} \mu(F \cap E_j) = \sum_{j=1}^{+\infty} \mu(F \cap E_{2j}) + \sum_{j=1}^{+\infty} \mu(F \cap E_{2j-1})$$
$$= \mu\left(\bigcup_{j=1}^{+\infty} F \cap E_{2j}\right) + \mu\left(\bigcup_{j=1}^{+\infty} F \cap E_{2j-1}\right) \leq 2\mu(F) < +\infty,$$

which concludes the proof of (5) and of Theorem 2. □

Terminology

Remark 4 (Outer measures)**.** *If X is a set and if the function $\mu : \mathcal{P}(X) \to [0, +\infty]$ defined on all subsets of X satisfies*

(a) $\mu(\emptyset) = 0$;
(b) *if* $E \subset F$, *then* $\mu(E) \leq \mu(F)$;
(c) *if* $\{E_i\}_{i \geq 1}$ *is a countable family of subsets of* X, *then*

$$\mu(E) \leq \sum_{i=1}^{+\infty} \mu(E_i) \qquad \text{where} \qquad E = \bigcup_{i=1}^{+\infty} E_i\,,$$

then we will say that μ is an **outer measure** *on X.*

Remark 5 (σ-algebra)**.** *Let X be a given set. If $\mathcal{A}$ is a family of subsets of X with the following properties*

- $\emptyset \in \mathcal{A}$ *and* $X \in \mathcal{A}$;
- *if* $E \in \mathcal{A}$, *then also* $X \setminus E \in \mathcal{A}$;
- *if* $E_1, E_2 \in \mathcal{A}$, *then also* $E_2 \cap E_2,\ E_1 \cup E_2,\ E_1 \setminus E_2,\ E_2 \setminus E_1 \in \mathcal{A}$;
- *if* $\{E_i\}_{i \geq 1}$ *is a family of sets such that* $E_i \in \mathcal{A}$, *then*

$$\bigcap_{i=1}^{+\infty} E_i \in \mathcal{A} \qquad \text{and} \qquad \bigcup_{i=1}^{+\infty} E_i \in \mathcal{A}\ ,$$

then we will say that $\mathcal{A}$ is a σ**-algebra**.

Remark 6 (Borel sets)**.** *Let X be a topological space. The family of* **Borel sets** *of on X is the smallest family $\mathcal{A}$ of subsets of X with the following properties:*

- $\mathcal{A}$ *contains all open subsets of* X;
- $\mathcal{A}$ *is a* σ*-algebra.*

Remark 7 (σ-additive measures)**.** *Let X be a given set and let $\mathcal{A}$ be a σ-algebra of subsets of X.*
If $\mu : \mathcal{A} \to [0, +\infty]$ is a function such that

- $\mu(\emptyset) = 0$;

- *if* $E_1, E_2 \in \mathcal{A}$ *are disjoint sets, then*
$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2);$$
- $\{E_i\}_{i\geq 1}$ *is a countable family of disjoint sets* $E_i \in \mathcal{A}$*, then*
$$\mu(E) = \sum_{i=1}^{+\infty} \mu(E_i) \qquad \textit{where} \qquad E = \bigcup_{i=1}^{+\infty} E_i\,,$$

then we will say that μ *is a* **(σ-additive) measure** *on* $\mathcal{A}$.

Remark 8 (Measurable sets)**.** *Let* X *be a given set and let* $\mu : \mathcal{P}(X) \to [0, +\infty]$ *be an outer measure on* X*. We will say that a set* $E \subset X$ *is* μ**-measurable** *if*
$$\mu(F) = \mu(F \cap E) + \mu(F \setminus E) \quad \textit{for all} \quad F \subset X.$$

Remark 9 (Borel measures)**.** *Let* X *be a given set and let* $\mu : \mathcal{P}(X) \to [0, +\infty]$ *be an outer measure on* X*. We will say that* μ *is a* **Borel measure***, if the* σ*-algebra of* μ*-measurable sets contains all Borel sets.*

In view of the above terminology, Theorem 1 can be restated as follows:

Theorem 10. *Let* X *be a given set. Let* $\mu : \mathcal{P}(X) \to [0, +\infty]$ *be an outer measure and let* $\mathcal{A}$ *be the family of all* μ*-measurable subsets of* X*. Then,* $\mathcal{A}$ *is a* σ*-algebra and* μ *is a* σ*-additive measure on* $\mathcal{A}$*.*