Hausdorff measures

Definition of $\mathcal{H}^s_\delta$

Definition 1 (Diameter of a set)**.** *Let $d \geq 1$. For any set $A \subset \mathbb{R}^d$ we define the diameter of A as*

$$\mathrm{diam}(A) = \sup\Big\{|x-y| \ : \ x, y \in A\Big\}.$$

Remark 2. *Notice that if $\overline{A}$ is the closure of A, then $\mathrm{diam}(A) = \mathrm{diam}(\overline{A})$.*

Definition 3 (The constants ω_s)**.** *For any $d \geq 0$ we define the constant ω_s as follows:*

- $\omega_0 = 1$;
- *if $n \in \mathbb{N}$, $n \geq 1$, then ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$;*
- *if $s \in \mathbb{R}$, $s > 0$, then we set:*

$$\omega_s := \pi^{s/2}\left(\int_0^{+\infty} t^{s/2}\, e^{-t}\, dt\right)^{-1}.$$

Definition 4 (Hausdorff measure $\mathcal{H}^s_\delta$)**.** *Let $d \geq 1$, $\delta > 0$ and $A \subset \mathbb{R}^d$ be any subset of $\mathbb{R}^d$.*

- *For any $s > 0$, we define*

$$\mathcal{H}^s_\delta(A) := \inf\Big\{\sum_{i=1}^{+\infty} \omega_s \left(\frac{\mathrm{diam}(U_i)}{2}\right)^s \ : \ U_i \subset \mathbb{R}^d,\ \bigcup_{i=1}^{\infty} U_i \supset A\Big\},$$

  where the infimum is taken among all families $\{U_i\}_{i\geq 1}$ of subsets of $\mathbb{R}^d$ that cover A.
- *When $s = 0$, we define*

$$\mathcal{H}^0_\delta(A) := \inf\Big\{\sum_{i=1}^{+\infty} \chi(U_i) \ : \ U_i \subset \mathbb{R}^d,\ \bigcup_{i=1}^{\infty} U_i \supset A\Big\},$$

  where the infimum is taken among all families $\{U_i\}_{i\geq 1}$ of subsets of $\mathbb{R}^d$ that cover A, and where for any set $U \subset \mathbb{R}^d$,

$$\chi(U) := \begin{cases} 1 & \text{if} \quad U \neq \emptyset \\ 0 & \text{if} \quad U = \emptyset. \end{cases}$$

Equivalent definitions of $\mathcal{H}^s_\delta$

It is immediate to check that one can work with coverings with closed sets or with subsets of A. We give the precise statements below; the proofs are an immediate consequence of the definition of $\mathcal{H}^s_\delta$.

Proposition 5 (Definition of $\mathcal{H}^s_\delta$ by covering with closed sets)**.** *Let $d \geq 1$ and $\delta > 0$. Let $A \subset \mathbb{R}^d$ be any subset of $\mathbb{R}^d$.*

- *For any $s > 0$, we have*

$$\mathcal{H}^s_\delta(A) := \inf\Big\{\sum_{i=1}^{+\infty} \omega_s \left(\frac{\mathrm{diam}(C_i)}{2}\right)^s \ : \ C_i \subset \mathbb{R}^d,\ C_i \ \textit{closed},\ \bigcup_{i=1}^{\infty} C_i \supset A\Big\},$$

  where the infimum is taken among all families $\{C_i\}_{i\geq 1}$ of closed subsets of $\mathbb{R}^d$ that cover A.
- *When $s = 0$, we have*

$$\mathcal{H}^0_\delta(A) := \inf\Big\{\sum_{i=1}^{+\infty} \chi(C_i) \ : \ C_i \subset \mathbb{R}^d,\ C_i \ \textit{closed},\ \bigcup_{i=1}^{\infty} C_i \supset A\Big\},$$

  where the infimum is taken among all families $\{C_i\}_{i\geq 1}$ of closed subsets of $\mathbb{R}^d$ that cover A.

Proposition 6 (Definition of $\mathcal{H}^s_\delta$ by covering with subsets)**.** *Let $d \geq 1$ and $\delta > 0$. Let $A \subset \mathbb{R}^d$ be any subset of $\mathbb{R}^d$.*

- *For any $s > 0$, we have*
$$\mathcal{H}^s_\delta(A) := \inf\Big\{\sum_{i=1}^{+\infty} \omega_s \left(\frac{\operatorname{diam}(U_i)}{2}\right)^s \ :\ U_i \subset A,\ \bigcup_{i=1}^{\infty} U_i \supset A\Big\},$$
where the infimum is taken among all families $\{U_i\}_{i\geq 1}$ of subsets of $A \subset \mathbb{R}^d$ that cover A.
- *When $s = 0$, we have*
$$\mathcal{H}^0_\delta(A) := \inf\Big\{\sum_{i=1}^{+\infty} \chi(U_i) \ :\ U_i \subset A,\ \bigcup_{i=1}^{\infty} U_i \supset A\Big\},$$
where the infimum is taken among all families $\{U_i\}_{i\geq 1}$ of subsets of A that cover A.

Basic properties of $\mathcal{H}^s_\delta$

Proposition 7 (Monotonicity with respect to δ)**.** *Let $d \geq 1$ and $A \subset \mathbb{R}^d$ be any subset of $\mathbb{R}^d$. Then, for any $s \geq 0$ we have that*
$$0 < \delta_1 < \delta_2 \quad \Rightarrow \quad \mathcal{H}^s_{\delta_1}(A) \geq \mathcal{H}^s_{\delta_2}(A).$$

Proof. Follows immediately from the definition as the infimum in $\mathcal{H}^s_{\delta_2}(A)$ is taken over a larger class of coverings with respect to the infimum in $\mathcal{H}^s_{\delta_1}(A)$. □

Proposition 8 (Monotonicity with respect to the set inclusion)**.** *Let $d \geq 1$, $s \geq 0$ and $\delta > 0$. Then, for any couple of sets $A_1 \subset \mathbb{R}^d$, $A_2 \subset \mathbb{R}^d$ we have that*
$$A_1 \subset A_2 \quad \Rightarrow \quad \mathcal{H}^s_\delta(A_1) \leq \mathcal{H}^s_\delta(A_2).$$

Proof. This follows since any covering $\{U_i\}_{i\geq 1}$ of A_2 is also a covering of A_1, so the infimum in $\mathcal{H}^s_\delta(A_1)$ is taken over a larger class of coverings than the infimum in $\mathcal{H}^s_{\delta_1}(A_2)$. □

Proposition 9 (σ-subadditivity of $\mathcal{H}^s_\delta$)**.** *Let $d \geq 1$ and let $\{A_k\}_{k\geq 1}$, be a countable family of subset of $\mathbb{R}^d$. Then, for any $s \geq 0$ and any $\delta > 0$ we have that*
$$\mathcal{H}^s_\delta(A) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s_\delta(A_k) \qquad \textit{where} \qquad A := \bigcup_{k=1}^{+\infty} A_k.$$

Proof. Suppose that $\mathcal{H}^s_\delta(A_k) < +\infty$ for all k. Fix $\varepsilon > 0$. For any $k \geq 1$ we can find a covering $\{U_i^k\}_{i\geq 1}$ of A_k with sets of diameter $\operatorname{diam}(U_i^k) \leq \delta$ such that
$$\mathcal{H}^s_\delta(A_k) \leq \sum_{i=1}^{+\infty} \omega_s \left(\frac{\operatorname{diam}(U_i^k)}{2}\right)^s \leq \mathcal{H}^s_\delta(A_k) + \frac{\varepsilon}{2^k}.$$
Then, the family
$$\{U_i^k\}_{i\geq 1, k\geq 1}$$
is a countable covering of A with sets of diameter at most δ. By the defintion of $\mathcal{H}^s_\delta(A)$ we have
$$\begin{aligned}\mathcal{H}^s_\delta(A) &\leq \sum_{k=1}^{+\infty}\sum_{i=1}^{+\infty} \omega_s \left(\frac{\operatorname{diam}(U_i^k)}{2}\right)^s \\ &\leq \sum_{k=1}^{+\infty}\Big(\mathcal{H}^s_\delta(A_k) + \frac{\varepsilon}{2^k}\Big) = \varepsilon + \sum_{k=1}^{+\infty} \mathcal{H}^s_\delta(A_k).\end{aligned}$$
Since $\varepsilon > 0$ is arbitrary, we get the claim. □

Proposition 10 (Additivity of $\mathcal{H}^s_\delta$ on distant sets)**.** *Let $d \geq 1$, $s \geq 1$. Suppose that A_1 and A_2 are disjoint sets in $\mathbb{R}^d$ lying on a positive distance $\rho > 0$ from each other, where*

$$\rho := \inf\Big\{|x_1 - x_2| \ : \ x_1 \in A_1, \ x_2 \in A_2\Big\}.$$

Then

$$\mathcal{H}^s_\delta(A_1 \cup A_2) = \mathcal{H}^s_\delta(A_1) + \mathcal{H}^s_\delta(A_2) \qquad \textit{for all} \qquad \delta \in (0, \rho).$$

Proof. Thanks to Proposition 9 we have that

$$\mathcal{H}^s_\delta(A_1 \cup A_2) \leq \mathcal{H}^s_\delta(A_1) + \mathcal{H}^s_\delta(A_2) \qquad \text{for all} \qquad \delta > 0.$$

In what follows to prove the opposite inequality for $\delta \in (0, \rho)$. Notice that it is sufficient to consider the case $\mathcal{H}^s_\delta(A_1 \cup A_2) < +\infty$. For every $\varepsilon > 0$ we can find a covering $\{U_i\}_{i\geq 1}$ of $A_1 \cup A_2$ of sets of diameter at most $\delta > 0$ such that

$$\mathcal{H}^s_\delta(A_1 \cup A_2) \leq \sum_{i=1}^{+\infty} \omega_s \left(\frac{\mathrm{diam}(U_i)}{2}\right)^s \leq \mathcal{H}^s_\delta(A_1 \cup A_2) + \varepsilon.$$

Notice that, up to eliminating some sets from $\{U_i\}_{i\geq 1}$, we can suppose that

$$U_i \cap (A_1 \cup A_2) \neq \emptyset \quad \text{for all} \quad i \geq 1.$$

Since the diameter of each U_i is smaller than the distance between A_1 and A_2, we have that either

$$U_i \cap A_1 \neq \emptyset \quad \text{and} \quad U_i \cap A_2 = \emptyset,$$

or

$$U_i \cap A_1 = \emptyset \quad \text{and} \quad U_i \cap A_2 \neq \emptyset.$$

We can divide the family $\{U_i\}_{i\geq 1}$ in two distinct families $\{U^1_j\}_{j\geq 1}$ and $\{U^2_k\}_{k\geq 1}$ with the property that

$$U^1_j \cap A_1 \neq \emptyset \quad \text{and} \quad U^1_j \cap A_2 = \emptyset \quad \text{for all} \quad j \geq 1,$$
$$U^2_k \cap A_1 = \emptyset \quad \text{and} \quad U^2_k \cap A_2 \neq \emptyset \quad \text{for all} \quad k \geq 1.$$

By construction $\{U^1_j\}_{j\geq 1}$ is a covering for A_1 and $\{U^2_k\}_{k\geq 1}$ is a covering for A_2. Thus

$$\begin{aligned}\mathcal{H}^s_\delta(A_1) + \mathcal{H}^s_\delta(A_2) &\leq \sum_{j=1}^{+\infty} \omega_s \left(\frac{\mathrm{diam}(U^1_j)}{2}\right)^s + \sum_{k=1}^{+\infty} \omega_s \left(\frac{\mathrm{diam}(U^2_k)}{2}\right)^s \\ &= \sum_{i=1}^{+\infty} \omega_s \left(\frac{\mathrm{diam}(U_i)}{2}\right)^s = \varepsilon + \mathcal{H}^s_\delta(A_1 \cup A_2).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get the claim. $\square$

Proposition 11 (Monotonicity in s)**.** *Let $\delta > 0$, $d \geq 1$ and A be any set in $\mathbb{R}^d$. Then, for any $0 \leq s < t$ we have the estimate*

$$\mathcal{H}^t_\delta(A) \leq \frac{\omega_t}{2^{t-s}\omega_s} \delta^{t-s} \mathcal{H}^{s_1}_\delta(A).$$

Proof. Suppose that $s > 0$, the case $s = 0$ being analogous.

For any $U \subset \mathbb{R}^d$ of diameter at most δ we have

$$\left(\frac{\mathrm{diam}(U)}{2}\right)^t \leq \left(\frac{\delta}{2}\right)^{t-s} \left(\frac{\mathrm{diam}(U)}{2}\right)^s.$$

Suppose now that $\{U_i\}_{i\geq 1}$ is any covering of A with sets of diameter at most δ. Then

$$\sum_{i=1}^{+\infty} \omega_t \left(\frac{\mathrm{diam}(U)}{2}\right)^t \leq \frac{\omega_t}{2^{t-s}\omega_s} \delta^{t-s} \sum_{i=1}^{+\infty} \omega_s \left(\frac{\mathrm{diam}(U)}{2}\right)^s.$$

Taking the infimum on both sides, we get the claim. $\square$

Definition of $\mathcal{H}^s$

Let $d \geq 1$, $s \geq 0$ and A be a subset of $\mathbb{R}^d$. Then, thanks to Proposition 7, we have that the function

$$(0 + \infty) \ni \delta \mapsto \mathcal{H}^s_\delta(A).$$

is monotone decreasing in δ. Then, the limit as $\delta \to 0$ exists and

$$\lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

This allows to give the following definition.

Definition 12 (The Hausdorff measure $\mathcal{H}^s$)**.** *Let $d \geq 1$ and $s \geq 0$. Then, for any set $A \subset \mathbb{R}^d$ we define*

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

Basic properties of $\mathcal{H}^s$

Proposition 13 (Monotonicity with respect to the set inclusion)**.** *Let $d \geq 1$ and $s \geq 0$. Then, for any couple of sets $A_1 \subset \mathbb{R}^d$, $A_2 \subset \mathbb{R}^d$ we have that*

$$A_1 \subset A_2 \quad \Rightarrow \quad \mathcal{H}^s(A_1) \leq \mathcal{H}^s(A_2).$$

Proof. Let $A_1 \subset A_2$. By Proposition 8, we have that

$$\mathcal{H}^s_\delta(A_1) \leq \mathcal{H}^s_\delta(A_2) \quad \text{for all} \quad \delta > 0.$$

Taking the limit as $\delta \to 0$, we get the claim. □

Proposition 14 (σ-subadditivity of $\mathcal{H}^s$)**.** *Let $d \geq 1$ and let $\{A_k\}_{k \geq 1}$, be a countable family of subset of $\mathbb{R}^d$. Then, for any $s \geq 0$ and any $\delta > 0$ we have that*

$$\mathcal{H}^s(A) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s(A_k) \qquad \textit{where} \qquad A := \bigcup_{k=1}^{+\infty} A_k.$$

Proof. Thanks to Proposition 9 we have that for any $\delta > 0$ it holds

$$\mathcal{H}^s_\delta(A) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s_\delta(A_k).$$

Now, since for each $k \geq 1$ it holds,

$$\mathcal{H}^s_\delta(A_k) \leq \mathcal{H}^s(A_k),$$

we get that

$$\mathcal{H}^s_\delta(A) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s(A_k).$$

Taking the limit as $\delta \to 0$, we get the claim. □

Proposition 15 (Additivity of $\mathcal{H}^s$ on distant sets)**.** *Let $d \geq 1$, $s \geq 1$. Suppose that A_1 and A_2 are disjoint sets in $\mathbb{R}^d$ lying on a positive distance from each other, that is,*

$$\inf \Big\{ |x_1 - x_2| \ : \ x_1 \in A_1, \ x_2 \in A_2 \Big\} > 0.$$

Then

$$\mathcal{H}^s(A_1 \cup A_2) = \mathcal{H}^s(A_1) + \mathcal{H}^s(A_2).$$

Proof. Set

$$\rho := \inf \Big\{ |x_1 - x_2| \ : \ x_1 \in A_1, \ x_2 \in A_2 \Big\}.$$

By hypothesis $\rho > 0$. Thanks to Proposition 10 we have that

$$\mathcal{H}^s_\delta(A_1 \cup A_2) = \mathcal{H}^s_\delta(A_1) + \mathcal{H}^s_\delta(A_2) \qquad \text{for all} \qquad \delta \in (0, \rho).$$

Taking the limit as $\delta \to 0$, we get the claim. □

The counting measure $\mathcal{H}^0$

In the next proposition we show that $\mathcal{H}^0$ is the measure that counts the number of points in a set, that is, $\mathcal{H}^0(A)$ is precisely the cardinality of the set A.

Proposition 16. *In any dimension $d \geq 1$ and for any subset $A \subset \mathbb{R}^d$, we have that*

$$\mathcal{H}^0(A) = \begin{cases} n & \text{if } A \text{ consists of exactly } n \text{ distinct points,} \\ +\infty & \text{if } A \text{ contains infinitely many distinct points.} \end{cases}$$

Proof. The proof follows from the following two claims.

Claim 1. If A consists of at most n points $x_1, \dots, x_n$, then $\mathcal{H}^0(A) \leq n$.

The family $U_i := \{x_i\}$, $i = 1, \dots, n$ is a covering of A. Since $\operatorname{diam}(U_i) = 0$, we have that

$$\mathcal{H}^0_\delta(A) \leq n \quad \text{for every} \quad \delta > 0.$$

Passing to the limit as $\delta \to 0$, we get the claim.

Claim 2. If A contains m distinct points $y_1, \dots, y_m$, then $\mathcal{H}^0(A) \geq m$.

Since all the points $y_1, \dots, y_m$ are distinct, we have that

$$\rho := \inf\Big\{|y_i - y_j| \ : \ i, j \in \{1, \dots, m\}\Big\} > 0.$$

Suppose now that $\delta \in (0, \rho)$ and that $\{U_k\}_{k\geq 1}$ is a covering of A of sets of diameter at most δ. Then, each of the sets U_k can contain at most one of the points y_i. Thus, the covering $\{U_k\}_{k\geq 1}$ consists of at least m distinct non-empty sets. By the definition of $\mathcal{H}^0_\delta$, we have that

$$\mathcal{H}^s_\delta(A) \geq m \quad \text{for any} \quad \delta < \rho.$$

Passing to the limit as $\delta > 0$ we get the claim. $\square$

The measure $\mathcal{H}^s$ when $s > d$

In the next proposition we show that when $s > d$ the measure $\mathcal{H}^s$ in $\mathbb{R}^d$ is trivial.

Proposition 17. *Let $d \geq 1$ and $A \subset \mathbb{R}^d$. Then,*

$$\mathcal{H}^s(A) = 0 \quad \textit{for any} \quad s > d.$$

Proof. Notice that, thanks to the monotonicity with respect to the inclusion (Proposition 13) and the σ-subadditivity (Proposition 14), it is sufficient to prove the claim for cubes $Q_R = [-R, R]^d$, $R > 0$. Indeed,

$$\mathcal{H}^s(A) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s(A \cap Q_k) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s(Q_k).$$

Fix $R > 0$. For any $n > 0$, we consider coverings $\{U_i^n\}_{i=1,\dots,2^{nd}}$ of the cube Q_R with 2^{nd} cubes of side $2^{-n}(2R)$. We notice that by construction

$$\operatorname{diam}(U_i^n) = \operatorname{diam}\Big([-2^{-n}(2R), 2^{-n}(2R)]^d\Big) = \sqrt{d}\, 2^{-n}(2R).$$

Then, fixed δ, we get that for any n such that

$$2^{-n} < \frac{\delta}{2R\sqrt{d}},$$

we have that

$$\mathcal{H}^s_\delta(Q_R) \leq \sum_{i=1}^{2^{nd}} \omega_s \left(\frac{\operatorname{diam}(U_i)}{2}\right)^s = \sum_{i=1}^{2^{nd}} \omega_s \left(\sqrt{d}\, R\, 2^{-n}\right)^s = d^{s/2}\, R^s\, 2^{-n(s-d)}.$$

Taking the limit as $n \to +\infty$, we get that

$$\mathcal{H}^s_\delta(Q_R) = 0 \quad \text{for each} \quad \delta > 0.$$

As a consequence $\mathcal{H}^s(Q_R) = 0$. $\square$

Hausdorff dimension

Proposition 18. *Let $d \geq 1$ and A be any set in $\mathbb{R}^d$.*
Then,the following holds:

- *if $s \geq 0$ and $\mathcal{H}^s(A) < +\infty$, then*
$$\mathcal{H}^t(A) = 0 \quad \text{for any} \quad t > s;$$
- *if $t > 0$ and $\mathcal{H}^t(A) > 0$, then*
$$\mathcal{H}^s(A) = +\infty \quad \text{for any} \quad s < t.$$

Proof. We only need to prove the first claim, the second one being an immediate corollary.
Let $s \geq 0$ and $t > s$. By Proposition 7, for any $\delta > 0$, we have
$$\mathcal{H}^t_\delta(A) \leq \frac{\omega_t}{2^{t-s}\omega_s}\delta^{t-s}\mathcal{H}^s_\delta(A).$$
On the other hand, by the definition of $\mathcal{H}^s$ we have
$$\mathcal{H}^s_\delta(A) \leq \mathcal{H}^s(A),$$
so
$$\mathcal{H}^t_\delta(A) \leq \frac{\omega_t}{2^{t-s}\omega_s}\delta^{t-s}\mathcal{H}^s(A).$$
Now, taking the limit as $\delta \to 0$ and using that $\mathcal{H}^s(A) < +\infty$, we get that $\mathcal{H}^t(A) = 0$. □

Definition 19 (Hausdorff dimension)**.** *Let $d \geq 1$ and let A be a subset of $\mathbb{R}^d$.*
We define the Hausdorff dimension of A as
$$\dim_{\mathcal{H}}(A) := \inf\Big\{s > 0 \ : \ \mathcal{H}^s(A) = 0\Big\}.$$

$\mathcal{H}^s$ are Borel measures on $\mathbb{R}^d$

Theorem 20. *Let $d \geq 1$ and $s \in (0, d]$. Define the family $\mathcal{A}_s$ of subset of $\mathbb{R}^d$ as follows: we say that $E \in \mathcal{A}_s$, if*
$$\mathcal{H}^s(F) = \mathcal{H}^s(F \cap E) + \mathcal{H}^s(F \setminus E) \quad \text{for all} \quad F \subset \mathbb{R}^d.$$
Then, we have the following:

(i) *$\mathcal{A}_s$ is a σ-algebra:*
- *$\emptyset \in \mathcal{A}_s$ and $\mathbb{R}^d \in \mathcal{A}_s$;*
- *if $E \in \mathcal{A}_s$, then also $\mathbb{R}^d \setminus E \in \mathcal{A}_s$;*
- *if $\{E_i\}_{i\geq 1}$ is a family of sets such that $E_i \in \mathcal{A}_s$, then*
$$\bigcap_{i=1}^{+\infty} E_i \in \mathcal{A}_s \qquad \text{and} \qquad \bigcup_{i=1}^{+\infty} E_i \in \mathcal{A}_s\ .$$

(ii) *$\mathcal{A}_s$ contains all sets of zero measure and all Borel sets, precisely:*
- *if $\mathcal{H}^s(E) = 0$, then $E \in \mathcal{A}_s$;*
- *If E is open, then $E \in \mathcal{A}_s$;*
- *If E is closed, then $E \in \mathcal{A}_s$.*

(iii) *$\mathcal{H}^s$ is a σ-additive measure on $\mathcal{A}_s$, that is:*
- *$\mathcal{H}^s(E) \geq 0$ for all $E \in \mathcal{A}_s$;*
- *if $E_1, E_2 \in \mathcal{A}_s$ are disjoint sets, then*
$$\mathcal{H}^s(E_1 \cup E_2) = \mathcal{H}^s(E_1) + \mathcal{H}^s(E_2);$$
- *$\{E_i\}_{i\geq 1}$ is a countable family of disjoint sets $E_i \in \mathcal{A}_s$, then*
$$\mathcal{H}^s(E) = \sum_{i=1}^{+\infty} \mathcal{H}^s(E_i) \qquad \text{where} \qquad E = \bigcup_{i=1}^{+\infty} E_i\ .$$

Proof. The proof follows from Proposition 14, Proposition 14 and a general result for measures. □