Istituzioni di Analisi Matematica

Esame scritto

23 gennaio 2026

Non è consetito l'uso di telefoni cellulari, tablet, smartwatch
(né di altri dispositivi connessi),
né di calcolatrici, libri, dispense, appunti...

Exercice 1. *Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|_{\mathcal{B}}$ and let $\mathcal{B}'$ be the dual of $\mathcal{B}$ with norm $\|\cdot\|_{\mathcal{B}'}$. Suppose that V is a closed subspace of $\mathcal{B}$ with the following property:*

$$\text{for every } T \in \mathcal{B}' \text{ there is } v \in V \text{ such that}$$
$$\|v\|_{\mathcal{B}} \leq 1 \text{ and } |T(v)| = \|T\|_{\mathcal{B}'}.$$

Prove that $V = \mathcal{B}$.

Exercice 2.

(a) *Suppose that:*
- *u_n is a sequence in $L^2(\mathbb{R})$ converging weakly to $u \in L^2(\mathbb{R})$;*
- *v_n is a sequence in $L^2(\mathbb{R})$ converging strongly to $v \in L^2(\mathbb{R})$.*

Prove that $u_n v_n$ converges weakly in $L^1(\mathbb{R})$ to uv.

(b) *Suppose that:*
- *u_n is a sequence in $L^2(\mathbb{R})$ converging weakly to $u \in L^2(\mathbb{R})$;*
- *v_n is a sequence in $L^2(\mathbb{R})$ converging weakly to $v \in L^2(\mathbb{R})$.*

Is it true that $u_n v_n$ must converge weakly in $L^1(\mathbb{R})$ to uv?

Exercice 3. *Prove that there is a minimizer of the functional*

$$F(u) = \int_0^1 (u'(x))^2 \, dx - u(1),$$

among all functions $u \in W^{1,2}(0,1)$ such that $u(0) = 0$.

Exercice 4. *Let I be the interval $(0,1)$ and let Ω be the unit square $(0,1) \times (0,1)$ in $\mathbb{R}^2$. Let $u \in W^{1,2}(I)$ and $v \in W^{1,2}(I)$.*

(a) *Prove that the function $w(x,y) = u(x)v(y)$ is in $W^{1,2}(\Omega)$.*

(b) *Prove that, if $u \in W_0^{1,2}(I)$ and $v \in W_0^{1,2}(I)$, then $w \in W_0^{1,2}(\Omega)$.*